Research proposal

Localized oscillations in inhomogeneous and damped periodically forced oscillatory medium

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1. Background

Hearing is a profound phenomena that combines several distinct complex behaviors to orchestrate together toward the ability of interpretation of sound waves into neurosignals. As such, hearing triggers many studies from both medicinal and mechanistic (basic science) directions.

The sounds that we hear are collected in the outer ear, which then cause the eardrum to start vibrate. The vibration of the eardrum is transmitted to the three ossicles. While the ossicles function as an efficient transmitter of sound to the cochlea, they might also have an influence on the frequency that act on the cochlea, meaning that ossicles movement might cause the frequencies of disturbances entertaining the cochlea to be smaller than the frequencies of disturbances acting on the eardrum. In the cochlea, spatial localization of incoming disturbance leads to entertainment of sensory cells (hair cells) belonging to a specific zone in space, which then transmitted to the brain, where it is further processed as an incoming sound. Cochlea has a spiral (snail) tube shape, partitioned among almost its entire length into two channels of liquids by non-liquid zone. This zone is attached from the bottom to a membrane (the Basilar membrane) and consists of sensory cells spreading along the entire length. Localization of the incoming sound wave in the cochlea is expressed in localized vibration response of the basilar membrane, causing the excitation of specifics hair cells.

Measurements of the basilar membrane response of living specimens seems to reveal a shape of a propagating wave within a finite spatial portion in the cochlea [7], the displacement of the membrane is physically small as compared to the cochlea length. Importantly, the locations of those portions were found to be dependent on the frequency of the incoming sound. An illustration of the spatial profiles responses of the cochlea to different sound stimulations is given in Fig.1.

In addition, it was experimentally discovered that the ear emits spontaneously sounds at specific frequencies [6], the so called otoacoustic emissions. The latter suggest that the cochlea have an active oscillatory nature, and as such cochlea oscillations can be entrained by incoming sound waves. In fact cochlea as an active media has been suggested in various models [1,9].

The spatial response of the cochlea in the presence of a sound signal seems to obey a case of oscillatory-wave forms, characterized by stationary and localized envelopes (OSLE), in an oscillatory system subjected to periodic forcing. The response of the cochlea can be regarded as a 1D problem because of the large aspect ratio (length to width) of the cochlea. Spatially localized structures in 1D correspond to homoclinic orbits of the governing equations written as a dynamical system, where time is being replaced by space as an argument [2]. Studies of OSLE in periodically forced spatially extended systems are related mostly to oscillons which are spatially localized uniform oscillations, such as in the resonances 2:1 [2,3] and 1:1 [2] of the Forced-Coplex-Ginzburg-Landau (FCGL) equation. The FCGL equation is a normal form quation describing weak spatio-temporal modulations of the oscillation amplitude near the onset of oscillations. However, the case of the
The auditory system seems to be different than the systems of previous studies due to the spatial inhomogeneity of the physical characteristics. The basilar membrane at the apex of the human cochlea is more than five times as broad as at the base. Moreover, the basilar membrane is relatively thin and floppy at the apex of the cochlea but thicker and tauter toward the base. This inhomogeneity may have a major influence on the dynamics of such system, one might expect, for example, that it will affect the stationary properties of the localized solutions. Motivated by this observation we study OSLE in inhomogeneous periodically forced system.

There are some studies trying to capture some of the dynamical behaviors of the cochlea. However, most of those studies attempt to capture the otoacoustic emissions and signal amplification in the cochlea. In general they don’t deal with the pattern formation behavior aspects in the cochlea, as we intend to do in our work.

2. Research Goal

The main goal of the proposed research is to identify and study oscillatory-wave forms, characterized by stationary and localized envelopes (OSLE), in a periodically forced oscillatory medium with spatially dependent parameters.

To achieve this goal, we propose to study a canonical model for periodically forced oscillatory media, focusing on the effect of spatially dependent parameters. We will obtain the OSLE of traveling-wave forms, and analyze their dynamical properties, specifically, their structure and stability.
3. Research Methods

- Linear stability analysis of the homogeneous solutions: analytical computation of the growth rate for infinitesimal periodic perturbations about uniform solutions. Linear stability analysis of inhomogeneous solutions will be done by numerical stability analysis.

- In order to study the origin of the localized states solutions in the parameters space, we will use multiple time scale method to obtain amplitude equations that describe weak modulations of the solutions appears in the vicinity of a Hopf bifurcation point (uniform or finite wave-number type).

- Numerical integration based on finite difference techniques will be implemented for space-time exploration of the suggested model.

- Numerical study of steady state solutions (including in the co-moving frame) of partial differential equations by continuation method using AUTO package [8]. This methodology will allow us to obtain also the solutions that are unstable in the PDE context.

4. Preliminary Results: Forced Oscillations

4.1. 1:1 parametric forcing

The model that will be used throughout this work is the FitzHugh-Nagumo (FHN) model [4,5]. The model was originally proposed as a simplified description of spike generation in axons of nerve cells, and it is studied extensively in the framework of oscillatory systems. The FHN model is a non-dimensional phenomenological model which captures the major pattern formation features and is relatively simple to carry mathematical analysis. In particular, The FHN model exhibits an instability to oscillations (Hopf bifurcation) that is a critical ingredient for localized oscillations.

A periodic forcing will be added to the model to mimic the effect of incoming sound signal. The type of the forcing is not yet clear, but the existence of otoacoustic emissions, and the understanding that the cochlea is a self-autonomous oscillator, suggests that the forcing is of parametric type.

The general periodically forced FHN is:

\[
\begin{align*}
\partial_t u &= u - u^3 - v + D(x)\nabla^2 u + \gamma_a/\epsilon \cdot \cos(w_ft - kx) \\
\partial_t v &= \epsilon(x)(u - a_1(x)v) + a_0(x) + \delta(x)\nabla^2 v
\end{align*}
\]

Where \( \gamma_a = \gamma \) is a forcing of additive type, and \( \gamma_p = u \cdot \gamma \) is a forcing of a parametric type.

We wish as a start to obtain OSLE in the simplest model possible, thus we set: \( \delta = 0, a_0 = 0, D = \text{const}, k = 0 \). While the justification for these parameters comes only from simplicity, the \( k = 0 \) assumption can be presumably explained physically by the consideration of the typical sound frequencies that the ear can detect (20kHz - 200Hz), the length of the cochlea (33 mm [7]) and the sound’s propagating velocity (in air: 343 m/sec). In what follows, we shall confront this assumption with the fact that the ossicles might change the frequencies of the disturbances entertaining the cochlea to a lower values. We shall start by study the system under parametric forcing, which has an inherent advantage over the additive one, since it does not change the stability properties of the
trivial (zero) solution, while with the additive forcing the zero solution does not exist at all. The model under those simplification reads:

\[
\partial_t u = u - u^3 - v + D\nabla^2 u + \gamma \cdot u \cdot \cos(w_ft)
\]

\[
\partial_t v = \epsilon(x)(u - a_1(x)v)
\]

We study linear stability analysis of the trivial solution of the unforced system. Introducing an infinitesimally small periodic disturbance of the type:

\[
u(x,t) = \delta u(x,t) e^{ikx} e^{i\sigma t} + c.c.,
\]

\[
v(x,t) = \delta v(x,t) e^{ikx} e^{i\sigma t} + c.c.
\]

Where c.c. stands for complex conjugate.

Inserting the disturbance into the equations and keeping only linear terms, we obtain a dispersion relation equation describing the initial dynamics of the disturbance:

\[
\sigma_\pm = \frac{1 - a_1\epsilon - Dk^2 \pm \sqrt{(1 - a_1\epsilon - Dk^2)^2 - 4\epsilon(1 - a_1(1 - Dk^2))}}{2}
\]

\[
\sigma_\pm = \sigma_R \pm i\sigma_I \quad \sigma_R = \frac{1}{2}(1 - a_1\epsilon - Dk^2) \quad \sigma_I = \frac{1}{2} \sqrt{4\epsilon - (1 - Dk^2 + a_1\epsilon)^2}
\]

The real part of the dispersion relation represents the growth rate of mode k, and the imaginary part represents the oscillation frequency. The system is unstable to Hopf at \(\epsilon_c = \frac{1}{a_1}\), meaning that for \(\epsilon < \epsilon_c\) uniform oscillations start to grow.

4.2. Choice of parameters

We will consider here the situation where \(a_1 < 1\), where only one stationary solution exists - the trivial solution, this is because we consider the localized states to be a homoclinic orbits in space of the trivial solution. For the case of \(a_1 > 1\) the trivial solution will become unstable and two more stationary solutions will appear.

Cochlea’s profile shape will motivate the choice of the spatial functions for the parameters, without having a direct connection to the physical properties of the cochlea. To keep fidelity with the cochlea we restrict the spatial function of parameters to be monotonic and positive. At first we follow previous attempts that considered an exponentially decreasing function as a choice of parameters [9].

The length of our system will be taken as the length of the basilar membrane - 33 mm [7]. Because the spatial functions of the parameters will be chosen to fit dynamical behaviors in the cochlea, length value will have no major influence on the results, as been tested in practice by numerical simulations.

The existence of otoacoustic emissions implies that the system is in a situation of a critical oscillator, i.e., the system is below (albeit very near) the onset of oscillations. The parameter that controls the distance from the instability onset is \(\lambda = \frac{\omega_c^2}{\epsilon_c} = 1 - a_1\epsilon\). We set \(\lambda \sim -0.5 \equiv d_c\) for any point in space, where \(d_c\) is the constant value, i.e., system is below the onset in any point of the spatial domain (damped system).

One can observe that by setting \(a_1\epsilon\) to be approximately constant, the main parameter which define the natural frequency of the oscillations is \(\epsilon\). Inversely, one might observe that \(\epsilon\) is directly connected to the frequency of the system:

\[
\epsilon = w_0^2 + \frac{(1 + d_c)^2}{4}
\]
In the auditory system, higher frequencies of sound are detected at the left hand side of space, and lower frequency are detected on the right side, so we will set $\epsilon$ to be monotonically decreasing function of $x$, and $a_1$ to be monotonically increasing function.

Generally, the range of natural frequencies that should be chosen has to fit to the range of frequencies in the cochlea. Suggested range of frequencies in the cochlea would be the typical sound frequencies that the ear can detect (20kHz-200Hz), but for our preliminary demonstration of OSLE in 1:1 resonance we choose for simplicity a narrower range of frequencies: between 6 to 4 (our model has no units).

4.3. Numerical results

The simplified model will be studied numerically by solving initial value problem, using finite difference method and explicit Euler scheme for time integration, with Neumann or Dirichlet boundary conditions, setting the system’s initial values to be zero.

A OSLE of uniform oscillation with oscillating frequencies equal to the forcing frequency is then observed and plotted in Fig.2.

One can see from Fig.2, that initially the system behaves in some oscillatory way, but later on, only a specific spatial portion persists to oscillate.

This results shows the possibility of obtaining solutions that describe OSLE of uniform oscillations form in this model, where their location can be controlled by the forcing frequency. But the goal of our model is to receive profiles similar to those that appear in the cochlea, which are OSLE of traveling wave form solutions.

![Figure 2: Time-space behavior of FHN model subjected to parametric periodic forcing at different frequencies. The field u is plotted in time and space, with: $\gamma = 4$, Neumann boundary conditions, $D = 10^{-5}$, $\epsilon = 38.27 \cdot e^{-0.2136x}$, $a_1 = 0.0392 \cdot e^{0.2136x}$, and $x \in [0, 3.3]$. The natural frequencies calculated from the linear stability analysis presented in the upper bar and range from $w_0 \in [6, 4]$.](image)

Next we will study 3 modifications of the simplified model shown above, that might enable us to receive OSLE of traveling wave form:

1. Spatial dependent dispersion relation.
2. Traveling wave forcing.
3. Existence of finite-wave number oscillatory instability.

While the first two mechanisms are optional within the framework of the general forced FitzHugh-Nagumo model, the last mechanism will exist only with addition of another field [10,11,12] to the system, and thus it will be done as a last step.
5. Research Plan

We will continue with the attempts to obtain the spatially localized profiles that resemble those that emerge in the cochlea, by modify the simplified model. The modifications will then be:

1. Using non-constant diffusion coefficient $D$ in order to control the spatial dependence of the dispersion relation.

2. Setting the second diffusion coefficient $\delta$ to be finite, again in order to control the spatial dependence of the dispersion relation.

3. Setting the forcing to be traveling wave forcing ($k \neq 0$).

4. Adding another field to the system in order to create finite wave-length oscillatory instability.

After each modification of the model we will numerically explore the existence of OSLE of traveling wave form.

Once the OSLE of traveling wave form solutions will be obtained we will analyze their dynamical properties, specifically, their structure and stability.

References


