Schwarzschild metric and Friedmann equations from Newtonian gravitational collapse

Masters Thesis proposal

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March 20, 2015

Abstract

As is well known, the $0-0$ component of the Schwarzschild space can be obtained by the requirement that the geodesic of slowly moving particles match the Newtonian equation. Given this result, we show here that the remaining components can be obtained by requiring that the inside of a Newtonian ball of dust matched at a free falling radius with the external space determines that space to be Schwarzschild, if no pathologies exist. Also we are able to determine that the constant of integration that appears in the Newtonian Cosmology, coincides with the spatial curvature of the FLRW metric. These results are of interest at least in two respects, one from the point of view of its pedagogical value of teaching General Relativity without in fact using Einstein's equation and second, the fact that some results attributed to to General Relativity can be obtained without using General Relativity indicates that these results are more general than the particular dynamics specified by General Relativity.These are already shown by us and that the paper is attached with the proposal. In this the proposal, firstly, we are trying to generalize the asymptotic behaviour of a hedgehog to a non-asymptotically flat space-time . Secondly, We can introduce rotation into the solution by introducing $\phi' = \phi - \omega.t$ with a periodicity for the new angle between 0 and 2π .

Contents

1 Introduction

Many aspects of General Relativity can be derived without using Einsteins Equations (EE). For example, it is well known that the 0-0 component of Schwarzschild metric can be derived from the requirement correspondance with the Newtonian limit. As shown in ref 1 [1], on which this essay is based, the $g_{\bar{r}\bar{r}}$ component can be derived from matching the external space , assumed to be static, to an internal dust shell. This derivation is very different from the derivations for the $g_{\bar{r}\bar{r}}$ component explained in ref 2[2] and criticized in ref 3 [3]. We use the Newtonian Cosmology results, where a homogeneous and isotropic dust ball with co-moving dust particles can also shown to be studied.

Finally, the self consistency of the treatment produces the outcome that the constant of integration k can be interpreted as the spatial curvature in FRLW spacetime without using the EE .

2 Proposal and Plan

The main aim of this section is to show that for the geometric parameter defining the FRW space coincides with the Newtonian Energy in k found from the interpretation of the Newtonian cosmology and to demonstrate that $k = \kappa$.

We start with a 3-sphere by considering first an embedding four dimensional Euclidean Space with metric to finally obtain the Infinitesimal line element in FRW space as,

$$ds^2 = -dt^2 + R(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right) \tag{1}$$

We have assume that that we have a co-moving observer which satisfies $r = const$. Independently of that, in the FLRW space we can use everywhere (not just at the boundary) the barred radius $\bar{r} = R(t)r$, which means $r = \frac{\bar{r}}{R(t)}$. we will find $g_{\bar{r}\bar{r}}$ by the above-said transformation to get a relation between $g_{\bar{r}\bar{r}}$ with κ .Matching this with the same value of $g_{\bar{r}\bar{r}}$ from the previous section, we can infer that $k = \kappa$.

Here we have gone about this problem in the opposite way, showing that the matching of these two spaces imposes severe constraints that allows us to derive Schwarzschild space in the outside and determine that the Newtonian constant of integration k has to be the spatial curvature of the internal FLRW internal space, all of this without using Einstein's equations .

The implication of the integration constant k is that it implies negative energy bound system, positive energy system and zero energy free system for $k > 0, k < 0$ and $k = 0$ respectively [4]. We've already shown for k>0 how to prove that k=κ . We think we are untouched with the proof still now for

1. Case-1 , $k = 0$
2. Case-2, $k > 0$

We are also planning to introduce the cosmological constant into the picture. The geometry of space-time can also be changed from flat to non-flat. Therefore, we can have options of getting various result in near future.

We can try to deepen our knowledge on the following topics from ref 9 [9] :

1. The equations for the cosmological backgrounds and for the density contrast can be obtained from the Newtonian cosmology even in the presence of non-vanishing pressure.We have to modify the continuity equation and equation of density contarsy shall agree with the corresponding relativistic equation written in the synchronous and comoving gauge.So, we can easily introduce pressure in this Cosmological background to produce some further better result.

2. The paper [9] is missing a space-time interpretation of the solutions of Newton's laws for gravitationally interacting particles.We might try to find them properly even in presence of pressure.

There is an interesting related issue that has been found in the work of E. Guendelman & A. Rabinowitz [10] with our paper [1] that the metric they have used $ds^2 = --Mdt^2 + M^{-1}dr^2 + r^2 d\Omega^2$, where M is a constant different from 1, used to describe gravitational field of a Hedgehog is not asymptotically flat. In our case, before doing a special tunning of paremeters, indeed we get a non asymptotically flat solution with this asymptotic behavior. So, we think we can generalize our our paper if we generalize our solutions to include cases where there is gravitational collapse also in presence of a hedgehog. We think that this could be an exciting project to figure out.

We also can add rotation to the proposal, in order to consider a slowly rotating solution, consider the transformation $\phi' = \phi - \omega.t$ followed by the NEW assignment of periodicity $0 < \ phi' < 2\pi$ then we obtain a new rotating metric, notice that $\phi' = \phi - \omega t$ followed by the new assignment of periodicity $0 < \phi' < 2\pi$ is not a new coordinate tranformation, due to the new assignment of periodicity $0 < phi' < 2\pi$.

In addition we shall be trying to show some of the intersreting aspects :

1. To otain g_{00} in a fashion we got $g_{\bar{r}\bar{r}}$ in the paper [1] and to show that g_{00} of Weinberg is consistent with Scwarschild for a certain choice of the integration constant.

2. To show that the de Sitter space can be written as$k = 1, k = -1$ and $k = 0$ for FLRW spaces$ds^2 = -dt^2 + R(t)^2 \left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2 \right)$ with $R = e^{\lambda t}$ for $k = 0$, $R = \frac{1}{\lambda} \cosh \lambda t$ for $k = 1$ and $R = \frac{1}{\lambda} \sinh \lambda t$ for $k = -1$

3. We might try to prove that the Milne Universe, i.e $ds^2 = -dt^2 + R(t)^2 [\frac{dr^2}{1+r^2} + r^2 d\Omega^2]$ satisfy the EMPTY Einstein's equation.

In the meantime of making this research propsal, we have prepared an essay (which shall appear very soon at arxiv) from the original work [1] to compete in the prestigious Gravity Research Foundation essay competition 2015 , US .

3 Intermediate Work

In a sphere of arbitrary size with homogeneous and isotropic in spacetime will yield Friedmann's second equation without Cosmological constant. The proof is as follows.

The equation of motion for a test mass m located on the boundary of a sphere shell be described in terms of a homogeneous positive parameter $R(t)$, where the coordinate of each particle expands according to $a(t) = \textit{constant} \cdot R(t)$, where the constant depends on the particular particle therefore such equation reads

$$\ddot{a} = -\frac{G}{a^2}\left(\frac{4\pi}{3}a^3\rho\right) = -\frac{4\pi G}{3}a\rho. \tag{2}$$

which implies a similar equation for the universal expansion factor $R(t)$

$$\ddot{R} = -\frac{G}{R^2}\left(\frac{4\pi}{3}R^3\rho\right) = -\frac{4\pi G}{3}R\rho. \tag{3}$$

Basically, this corresponds to Friedmann's second equation without a cosmological constant Λ and zero pressure. As the linear dimensions scale by $R(t)$, all co-moving volumes should scale by $R(t)^3$, that is a $1/R^3$ dependency for the density, which dilutes the matter as the sphere expands.

For deriving a second equation, we first consider mass conservation within comoving sphere,

$$\frac{d}{dt}\left(\frac{4\pi}{3}R^3\rho\right) = 0, \tag{4}$$

where the internal mass M inside the sphere should be constant . By performing the derivative and simplifying one R, the equation gets

$$2R\dot{R}\rho + R\dot{R}\rho + R^2\dot{\rho} = 0.$$

The second term can be eliminated by (3) and after restoring derivatives the equation

$$\frac{d(\dot{R}^2)}{dt} = \frac{8\pi G}{3}\frac{d(\rho R^2)}{dt}$$

is obtained. Integration on both sides gives

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - \tilde{k},$$

and rewriting the arbitrary integration constant $\tilde{k}$ in a way to match the units $\tilde{k} \longrightarrow kc^2$ yields finally an equation, which corresponds to Friedmann's first equation in structure:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - k\left(\frac{c}{R}\right)^2 \tag{5}$$

As we all know the geodesic equation can be derived from the from the principle of least action of the particle trajectory, the action being the proper time along the trajectory of the particle in a certain given metric. Now, that in order to obtain the Newtonian non relativistic limit, the correct Newtonian force equation coincides with the geodesic equation provided by following equations

$$\frac{d^2x}{dt^2} = -\nabla\phi \quad where \quad \phi = -\frac{GM}{r} \tag{6}$$

and

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right) = -\left(1 + 2\phi\right) \tag{7}$$

The $g_{\bar{r}\bar{r}}$ of the metric in the outside static region in our case, has been the

"elusive component". This component has not been calculated using matching to a Newtonian cosmology previously, here we will show that this is possible[1]. In this section we will find $g_{\bar{r}\bar{r}}$ using the assumption that we have a co-moving observer satisfying $r = constant$. We will also assume that inside and at the boundary of the dust ball the radius evolves as $\bar{r} = rR(t)$, where $R(t)$ is determined by (5). We also assume that for $\bar{r} > rR(t)$ the motion is on a radial geodesic and the metric is of the form

$$ds^2 = -\left(1 - \frac{2GM}{\bar{r}}\right) d\bar{t}^2 + A(\bar{r})d\bar{r}^2 + \bar{r}^2 d\Omega^2 \tag{8}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{9}$$

we labelled the time $\bar{t}$ because it may not be the same coordinate as the t in the dust ball and the g_{tt} component was found in the previous section. Notice that we assume that the external metric is time independent.
A radially falling geodesic, meaning that $\theta = const$ and $\phi = const$, is fully described by the conservation of energy that results from that the metric outside is assumed to be static. The geodesics are derived from the action

$$S = \int d\sigma \sqrt{-\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} g_{\mu\nu}(x)} \tag{10}$$

The equation with respect to $\bar{t}$ is

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{\bar{t}}}\right) = 0 \quad where \quad \dot{\bar{t}} = \frac{d\bar{t}}{d\sigma} \tag{11}$$

This gives us

$$\gamma = \frac{\partial L}{\partial \dot{\bar{t}}} = \left(1 - \frac{2GM}{\bar{r}}\right)\frac{d\bar{t}}{d\tau} \tag{12}$$

where γ is constant and $d\tau$ is the proper time.
Notice that since the spatial coordinates in the space $dx^i = 0$ we get

$$d\tau = dt \tag{13}$$

giving us

$$d\tau^2 = \left(1 - \frac{2GM}{\bar{r}}\right) d\bar{t}^2 - A(\bar{r})d\bar{r}^2 \tag{14}$$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 = \left(1 - \frac{2GM}{\bar{r}}\right)\left(\frac{d\bar{t}}{dt}\right)^2 - A(\bar{r})\left(\frac{d\bar{r}}{dt}\right)^2 \tag{15}$$

Using (30) and (31) we obtain

$$\left(1-\frac{2GM}{\bar{r}}\right)^{-1}\gamma^2 - A(\bar{r})\left(\frac{d\bar{r}}{dt}\right)^2 = 1 \tag{16}$$

As we have seen, the consistency of the matching of the two spaces requires $\bar{r} = R(t)r$, furthermore, we assume that even the boundary of the dust shell free falls according to a co-moving observer, which means that the FLRW coordinate $r = constant$ and this allow then to solve for $A(\bar{r})$,

$$A(\bar{r}) = -\left(1 - \frac{\gamma^2}{\left(1-\frac{2GM}{\bar{r}}\right)}\right)\frac{1}{r^2\left(\frac{dR}{dt}\right)^2} \tag{17}$$

We now use (3) and get

$$A(\bar{r}) = -\left(1 - \frac{\gamma^2}{\left(1-\frac{2GM}{\bar{r}}\right)}\right)\frac{1}{r^2 k\left(\frac{1}{R}-1\right)} \tag{18}$$

Simplifying this and expressing in terms of $\bar{r}$ we get

$$A(\bar{r}) = \frac{1}{r^2}\frac{\gamma^2 - 1 + \frac{2GM}{\bar{r}}}{1-\frac{2GM}{\bar{r}}}\frac{1}{k\left(-1+\frac{r}{\bar{r}}\right)} \tag{19}$$

If we take the limit $\bar{r} \to \infty$, we see that $A(\bar{r}) \to -(\gamma^2-1)/kr^2$. Asymptotic flatness would require $A(\bar{r}) \to 1$.

The metric component $A(r)$ is free of singularities (real singularities, not coordinate singularities) and preserves its signature (one time and three spaces)and is asymptotically flat only if

$$\gamma^2 - 1 = -kr^2 \quad and \quad \frac{2GM}{\bar{r}} = \frac{r^2 k}{R} = \frac{kr^3}{\bar{r}} \Rightarrow k = \frac{2GM}{r^3} \tag{20}$$

Notice that above the condition,(20) $k = \frac{2GM}{r^3}$, when combined with the value of k, as given by (**??**), $k = \frac{8\pi G}{3}\rho(0)$ tells us the eminently reasonable relation,

$$M = \frac{4}{3}\pi\rho_0 r^3 \tag{21}$$

which indicates that this choice has a good physical basis. Finally, all of this gives us

$$A(\bar{r}) = \frac{1}{1-\frac{2GM}{\bar{r}}} \tag{22}$$

Reproducing the Schwarzschild spacetime.

4 Implications

In this paper[1] it has been found that matching a dust ball, whose dynamics is governed by the Newtonian cosmology equations, containing a constant of integration k coincides with the geometrical parameter κ that appears in FLRW used in the Newtonian Cosmology. These results are of interest at least in two respects, one from the fact that some results attributed to General Relativity can be obtained without using General Relativity indicates that these results are more general than the particular dynamics specified by General Relativity[6] and second from the point of view of its pedagogical value of teaching General Relativity without in fact using Einstein's equation. Finally, an interesting question is whether our derivation holds only in the weak field approximation or not. Notice that indeed in parts of our arguments we have used the weak field approximation, like when we derived the $0-0$ component of the metric, then on this basis, we derived the $r-r$ component by a process of matching to the internal collapsing ball of dust. But if we take the point of view that we trust the metric of the collapsing dust beyond the weak field approximation, the situation will be different, in this case our derivation could have validity beyond the weak field approximation, this is a question to be studied. One should also point out that in GR, as far as the post-Newtonian approximation is concerned, the corrections to the the $0-0$ component of the metric appear at the same other as the first corrections to the $r-r$ components, so it is in a sense puzzling that the $r-r$ component has been more elusive to find by a simple derivation, as compared to the case of the $0-0$ component.There is an interesting related issue that has been found in the work of E. Guendelman & A. Rabinowitz [10] with our paper [1] that the metric they have used $ds^2 = --Mdt^2 + M^{-1}dr^2 + r^2 d\Omega^2$, where M is a constant different from 1, used to describe gravitational field of a Hedgehog is not asymptotically flat. In our case, before doing a special tunning of paremeters, indeed we get a non asymptotically flat solution with this asymptotic behavior. So, we think we can generalize our our paper if we generalize our solutions to include cases where there is gravitational collapse also in presence of a hedgehog. We think that this could be an exciting project to figure out. We also can add rotation to the proposal, in order to consider a slowly rotating solution, consider the transformation $\phi' = \phi - \omega.t$ followed by the NEW assignment of periodicity $0 < \phi' < 2\pi$ then we obtain a new rotating metric, notice that $\phi' = \phi - \omega t$ followed by the new assignment of periodicity $0 < \phi' < 2\pi$ is not a new coordinate tranformation, due to the new assignment of periodicity $0 < \phi' < 2\pi$.

5 Acknowledgement

The author is deeply indebted to Prof. Eduardo Guendelman and Prof. Alex Gersten for their consistent support and helpful suggestion. Author also wants to show gratitute to Prof. David Eichler for bringing him at the Ben Gurion University of the Negev, Israel.

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