E2042: Pressure of an ideal gas in a box with gravitation

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The problem:

An ideal classical gas of N particles of mass m is in a container of height L which is in a gravitational field of a constant acceleration g. The gas is in uniform temperature T.

(a) Find the dependence $P(h)$ of the pressure on the height h.

(b) Find the partition function and the internal energy. Examine the limits $mgL \ll k_B T$ and $mgL \gg k_B T$.

(c) Consider an adiabatic atmosphere, i.e. the atmosphere has been formed by a constant entropy process in which T, μ, are not equilibrated, but $Pn^{-\gamma} = \text{const.}$, equilibrium is maintained within a layer at height h. Find $T(h)$ and $n(h)$ in terms of the density n_0 and temperature T_0 at $h = 0$.

Baruch's A19.

The solution:

(a) For a layer of height dh and area A in mechanical equilibrium:

$$A\left[P(h) - P(h+dh)\right] = n(h)mg \cdot Adh \tag{1}$$

leading to the differential equation:

$$dP = -nmg \cdot dh \tag{2}$$

Since that for an ideal gas $P = nT$, we can write:

$$\frac{dP}{P} = -\beta mg \cdot dh \qquad \Rightarrow \qquad P(h) = P_0\,\mathrm{e}^{-\beta mgh} \tag{3}$$

In order to find P_0, we integrate over $n(h)$ to get the total number of particles:

$$A\int_0^L n(h)dh = A\beta P_0 \int_0^L \mathrm{e}^{-\beta mgh} dh = \frac{AP_0}{mg}\left(1 - \mathrm{e}^{-\beta mgL}\right) = N \tag{4}$$

from which we get $P_0 = \frac{Nmg}{A}\left(1 - \mathrm{e}^{-\beta mgL}\right)^{-1}$.

(b) The one particle partition function is:

$$Z_1 = \frac{1}{(2\pi)^3}\int d^3p\, d^3x\, \mathrm{e}^{-\beta\left(\frac{p^2}{2m} + mgh\right)} = \frac{A}{\lambda_T^3}\int_0^L \mathrm{e}^{-\beta mgh} dh = \frac{A}{\lambda_T^3}\frac{1 - \mathrm{e}^{-\beta mgL}}{\beta mg} \tag{5}$$

The N-particles partition function is simply $\frac{1}{N!}Z_1^N$.The internal energy is given by:

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{N}{\beta}\left(\frac{5}{2} - \frac{\beta mgL}{\mathrm{e}^{\beta mgL} - 1}\right) \tag{6}$$

In the high tempratures limit $\beta mgL \ll 1$ the internal energy behaves as a $3D$ ideal gas:

$$E \approx \frac{N}{\beta}\left(\frac{5}{2} - \frac{\beta mgL}{1+\beta mgL - 1}\right) = \frac{3}{2}NT \tag{7}$$

as expected. Thus it is not surprising that we get uniform density:

$$n(h) = \beta P(h) = \frac{N}{V}\frac{\beta mgL\mathrm{e}^{-\beta mgh}}{1-\mathrm{e}^{-\beta mgL}} \approx \frac{N}{V} \tag{8}$$

In the low temperature limit $\beta mgL \gg 1$:

$$E \approx \frac{N}{\beta}\left(\frac{5}{2} - \beta mgL\,\mathrm{e}^{-\beta mgL}\right) \approx \frac{5}{2}NT \tag{9}$$

and unless $h = 0$, the density goes to zero:

$$n(h) \approx \frac{N}{V}\beta mgL\mathrm{e}^{-\beta mgh} \longrightarrow \frac{N}{A}\delta(h) \tag{10}$$

meaning that all particles are at the bottom of the container.

(c) We now have an adiabatic process where $P = Cn^{\gamma}$ with $\gamma > 1$. Within each layer equilibrium is maintained, so the usual state-equation of the ideal gas holds. For $h = 0$ we have:

$$P_0 = Cn_0^{\gamma} = n_0T_0 \qquad \Rightarrow \qquad C = T_0 n_0^{1-\gamma} \tag{11}$$

Using $dP = \gamma Cn^{\gamma-1}dn$, equation (2) now has the form:

$$\gamma Cn^{\gamma-1}dn = -nmg\cdot dh \qquad \Rightarrow \qquad \int_{n_0}^{n} n^{\gamma-2}dn = -\frac{mgh}{\gamma C} \tag{12}$$

We obtain:

$$n^{\gamma-1} = n_0^{\gamma-1} - (\gamma-1)\frac{mgh}{\gamma C} = n_0^{\gamma-1}\left(1 - \frac{\gamma-1}{\gamma}\cdot\frac{mgh}{n_0^{\gamma-1}C}\right) = n_0^{\gamma-1}\left(1-\frac{\gamma-1}{\gamma}\cdot\frac{mgh}{T_0}\right) \tag{13}$$

where we have used (11) in the last equality. We can now write:

$$n(h) = n_0\left(1 - \frac{\gamma-1}{\gamma}\cdot\frac{mgh}{T_0}\right)^{\frac{1}{\gamma-1}} \tag{14}$$

and the temperature is:

$$T(h) = Pn^{-1} = Cn^{\gamma-1} = T_0 - \left(\frac{\gamma-1}{\gamma}\right)mgh \tag{15}$$

We can see that both the temperature and the density decrease as we elevate and they vanish when we reach $h = \frac{\gamma T_0}{(\gamma-1)mg}$.