Solutions exam 2008

1. (a) With $\lambda = h/\sqrt{2\pi m k_B T}$, $\beta = 1/k_B T$,

$$Z = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N (\mathrm{e}^{\beta\mu B} + \mathrm{e}^{-\beta\mu B})^N = \mathrm{e}^{-\beta F}$$

$$S = \frac{\partial F}{\partial T} = S_{kin} + N\frac{\partial}{\partial T}(k_B T \ln 2\cosh(\beta\mu B))$$

The standard ideal gas result is

$$S_{kin}/N = k_B \ln \frac{V}{N\lambda^3} + \frac{5}{2}k_B$$

while the spin part is

$$S_{spin}/k_B N = \ln 2\cosh(\beta\mu B) - \beta\mu B \tanh(\beta\mu B)$$

In partcular this vanishes at $B \to \infty$ where only one state is allowed, while at B=0 it is $\ln 2$, corresponding to 2 states.

In an adiabatic process $S_{kin}(T_i) + S_{spin}(B, T_i) = S_{kin}(T_f) + S_{spin}(0, T_f)$. Therefore

$$S_{kin}(T_i) - S_{kin}(T_f) = k_B N \ln \frac{\lambda^3(T_f)}{\lambda^3(T_i)} = \frac{3}{2}k_B N \ln \frac{T_i}{T_f} = S_{spin}(0, T_f) - S_{spin}(B, T_i)$$

(b) At $B \to \infty$ $S_{spin}(T_f, B) = 0$, hence $\ln \frac{T_f}{T_i} = -\frac{2}{3}\ln 2$ so that $T_f = T_i/2^{2/3}$.

(c) In d dimensions $\lambda^3 \to \lambda^d$ and the spin has $2S+1$ states, hence $S_{spin}(0, T_f) = k_B N \ln(2S+1)$,

$$\frac{d}{2}\ln\frac{T_f}{T_i} = -\ln(2S+1)\,, \Rightarrow \qquad T_f = T_i/(2S+1)^{2/d}$$

2. (a) Counting the number of distinct ways of putting M particles in M locations, allowing for a^3 configurational integral and the usual λ^{-3} for momentum integrals, using $\zeta = \mathrm{e}^{\beta\mu}$,

$$\mathcal{L} = \sum_{N=0}^{M} \zeta^N \left(\frac{a^3}{\lambda^3}\right)^N \frac{M!}{N!(M-N)!} = (1 + \zeta\frac{a^3}{\lambda^3})^M$$

$$N = \zeta\frac{\partial \mathcal{L}}{\partial \zeta} = \zeta M \frac{\partial \ln(1+\zeta a^3/\lambda^3)}{\partial \zeta} = \zeta M \frac{a^3/\lambda^3}{1+\zeta a^3/\lambda^3}$$

$$n = \frac{N}{Ma^3} = \frac{\zeta/\lambda^3}{1+\zeta a^3/\lambda^3}$$

(b)

$$\ln \mathcal{L} = \beta PV = M \ln(1+\zeta a^3/\lambda^3)$$

$$na^3 = \frac{1}{\frac{1}{\zeta}\frac{\lambda^3}{a^3} + 1}, \qquad \Rightarrow \frac{1}{\zeta}\frac{\lambda^3}{a^3} = \frac{1}{na^3} - 1, \qquad \Rightarrow \zeta\frac{a^3}{\lambda^3} = \frac{na^3}{1-na^3}$$

$$Pa^3 = k_B T \ln(1 + \frac{na^3}{1-na^3}) = k_B T \ln \frac{1}{1-na^3}$$

At $n \to 0$ we get $P \to nk_BT$ the ideal gas. At $n \to \infty$ we get a singularity due to the excluded volume. Pressure diverges since particles cannot be added beyond the density $1/a^3$.

(c) Full lattice gas problem has nearest neighbor attarction that leads to a 1st order transition at $n < 1/a^3$. This attraction maps to J of the Ising model and that allows a 1st order phase transition.

3. (a) Photons have $\mu = 0$ and equilibrium for the reaction is $0 = \mu_+ + \mu_-$ and from neutrality $\mu_+ = \mu_- = 0$. Hence, with $\lambda = h/\sqrt{2\pi m_e k_B T}$

$$n_\pm = \frac{2}{V}\sum_p \frac{1}{\mathrm{e}^{\beta\sqrt{m_e^2c^4+p^2c^2}}+1} \approx \frac{2}{V}\sum_p \frac{1}{\mathrm{e}^{\beta m_e c^2+\beta p^2/2m_e}+1} = \frac{2}{\lambda^3} f_{3/2}(\mathrm{e}^{-\beta m_e c^2}) \approx \frac{2}{\lambda^3}\mathrm{e}^{-\beta m_e c^2}$$

(b) As above $\mu_\pm = 0$, now with $\lambda = h/\sqrt{2\pi m_\pi k_B T}$

$$n_\pm = \frac{1}{V}\sum_p \frac{1}{\mathrm{e}^{\beta\sqrt{m_\pi^2c^4+p^2c^2}}-1} \approx \frac{1}{V}\sum_p \frac{1}{\mathrm{e}^{\beta m_\pi c^2+\beta p^2/2m_\pi}-1} = \frac{1}{\lambda^3} g_{3/2}(\mathrm{e}^{-\beta m_\pi c^2}) \approx \frac{1}{\lambda^3}\mathrm{e}^{-\beta m_\pi c^2}$$

Np singularity at $p = 0$, hence no condensation. The possibility to transform $\pi^\pm$ into photons does not constrain the momentum sum on states. In particular at $T = 0$ the exponent $\mathrm{e}^{\beta\sqrt{m_e^2c^4+p^2c^2}}$ diverges even for $p = 0$, hence $n_\pm = 0$.

(c) Neutrality implies that μ_e for both $e^\pm$ is equal, and the same for μ_π. Equilibrium implies $\mu_e = \mu_\pi$ which we call now μ. n_e is the density for either $e^\pm$ and n_π is the density for either $\pi^\pm$. $\bar{n}$ is conserved in the reaction, so that

$$\frac{2}{V}\sum_p \frac{1}{\mathrm{e}^{\beta\sqrt{m_e^2c^4+p^2c^2}-\beta\mu}+1} + \frac{1}{V}\sum_p \frac{1}{\mathrm{e}^{\beta\sqrt{m_\pi^2c^4+p^2c^2}-\beta\mu}-1} + \langle n_0\rangle = \bar{n}$$

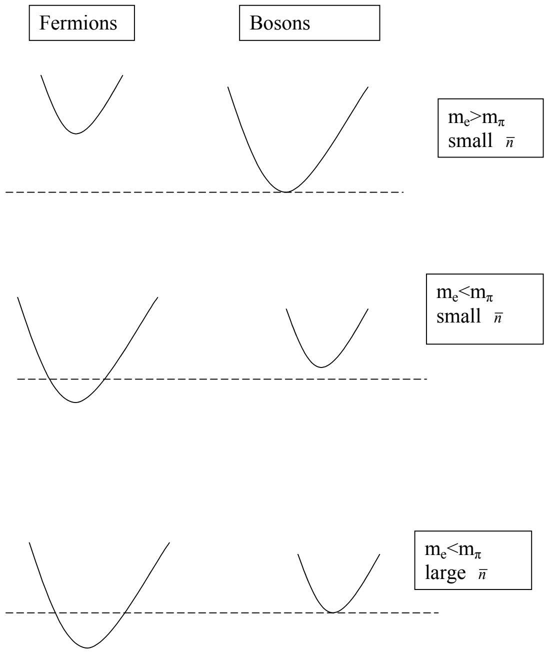
where $\langle n_0\rangle$ is the boson $p = 0$ occupation. Since boson occupation must be positive $\sqrt{m_\pi^2c^4+p^2c^2} - \mu \geq 0$ for all p, hence $\mu \leq m_\pi c^2$ at all T.

Consider now $T = 0$ and first $m_e > m_\pi$. Hence $\mu < m_e c^2$ and the fermion occupation factors vanish, $n_e = 0$. Hence all particles are $\pi^\pm$ in their ground state $p = 0$, i.e. the pions are in a Bose condensed phase (upper situation in figure, showing the energy bands as function of p_x, for example). If $m_e < m_\pi$ the Fermi energy μ may be located in between the minima of the two bands, $m_e c^2 < \mu < m_\pi c^2$, and then only fermions $e^\pm$ are present (2nd situation in figure). This situation occurs if $n_e = \bar{n}$ has a Fermi energy $\mu < m_\pi c^2$. The Fermi momentum p_F is determined by

$$m_e^2c^4 + p_F^2c^2 = \mu^2\,, \qquad \Rightarrow \qquad p_F = \sqrt{\mu^2 - m_e^2c^4}/c$$

Therefore

$$n_e = 2\int_{p<p_F} \frac{d^3p}{(2\pi\hbar)^3} = \frac{p_F^3}{3\pi^2\hbar^3} = \frac{1}{3\pi^2\hbar^3c^3}(\mu^2 - m_e^2c^4)^{3/2}$$



the condition for $n_\pi = 0$ is $\mu < m_\pi c^2$, hence

$$\bar{n} < \frac{c^3}{3\pi^2\hbar^3}(m_\pi^2 - m_e^2)^{3/2}$$

If $\bar{n} > \frac{c^3}{3\pi^2\hbar^3}(m_\pi^2 - m_e^2)^{3/2}$ then $\mu = m_\pi c^2$, at the $p = 0$ level of the pions (lower situation in figure). The fermions density is fixed and any increase in $\bar{n}$ goes into the pions that are bose condensed:

$$n_e = \frac{c^3}{3\pi^2\hbar^3}(m_\pi^2 - m_e^2)^{3/2}$$

$$n_\pi = \bar{n} - \frac{c^3}{3\pi^2\hbar^3}(m_\pi^2 - m_e^2)^{3/2}$$

4. (a) Since $0 = (\sum_i q_i)^2 = 2\sum_{i<j} q_i q_j + \sum_i q_i^2$ we have $\sum_{i<j} q_i q_j = -\frac{1}{2}2Nq^2 = -Nq^2$, hence

$$Z(L) = Z_{kin}\prod_i^{2N}\int_0^L d^2r_i e^{\beta\sum_{i<j} q_i q_j \ln|\mathbf{r}_i-\mathbf{r}_j|}$$

$$= Z_{kin}C^{-4N}\prod_i^{2N}\int_0^{CL} d^2r_i' e^{\beta\sum_{i<j} q_i q_j \ln|\mathbf{r}_i'-\mathbf{r}_j'|-\beta\sum_{i<j} q_i q_j \ln C}$$

$$\Rightarrow \qquad Z(L) = C^{-4N}C^{\beta q^2 N}Z(CL) = L^{4N-\beta q^2 N}Z(1) \sim A^{N(2-\frac{1}{2}\beta q^2)}$$

where $C = 1/L$ is chosen, and then $Z(1)$ is L independent.

(b) $P = -\frac{\partial F}{\partial A} = k_B T N(2 - \frac{1}{2}\beta q^2)/A$

A criterion for thermodynamic instability is $(\partial/P\partial A)_{T,N} > 0$. At $T < q^2/4$ this criterion is not satisfied indicating an instability. At low T the interaction dominates and pairs of $\pm$ charges tend to form neutral bound states. For fixed N the pairs are not allowed to annihilate, and to sustain bound states of finite size the model needs a short distance cutoff.

(c) $Z(1)$ includes N dependent factors from the momentum integrals $(\lambda^{-2})^{2N}$ for the 2N particles and the Gibbs correction $(1/N!)^2$ for the positive and negative ions separately. ($Z(1)$ contains additional N dependence from the interactions at short range $r_i < 1$ that are neglected).

$$Z(A,N) \sim \frac{1}{(N!)^2\lambda^{4N}}A^{N(2-\frac{1}{2}\beta q^2)}$$

$$F = k_B T[2N\ln\lambda^2 + 2N\ln N - 2N - N(2-\tfrac{1}{2}\beta q^2)\ln A] + \delta F$$

where δF is assumed to be N independent.

$$\mu = \partial F/\partial N = k_B T[2\ln\lambda^2 + 2\ln N - (2-\tfrac{1}{2}\beta q^2)\ln A]$$

$$e^{\beta\mu} = \lambda^4 N^2 A^{-2+\frac{1}{2}\beta q^2}, \qquad \Rightarrow N = \lambda^{-2}A^{1-\beta q^2/4}e^{\frac{1}{2}\beta\mu}$$

In the thermodynamic limit $A \to \infty$ we note that for a fixed μ at temperatures $T > T_c = q^2/4$ the system is a plasma with charges $N \to \infty$, while at $T < T_c$ the system is an insulator with no charges $N \to 0$, i.e. $\pm$ charges annihilate. Hence the proper way of viewing this phase transition is by fixing μ and allowing N to be variational.

5. (a) the circuit equation for $Q(t)$, adding a random $V(t)$, is

$$\frac{Q}{C} + R\dot{Q} + L\ddot{Q} = V_0 \cos\omega t + V(t)$$

After Fourier transform, using $\langle V(t)\rangle = 0$,

$$\langle Q(\omega)\rangle(\frac{1}{C} - i\omega R - L\omega^2) = \tfrac{1}{2}V_0\,, \qquad \Rightarrow \alpha_Q(\omega) = \frac{1}{\frac{1}{C} - i\omega R - L\omega^2}\,.$$

The dissipation rate is (lecture notes page 62)

$$\overline{\frac{dE}{dt}} = \tfrac{1}{2}\omega V_0^2 \mathrm{Im}\alpha_Q(\omega) = \tfrac{1}{2}V_0^2 \frac{\omega^2 R}{(L\omega^2 - \frac{1}{C})^2 + \omega^2 R^2}$$

(b)

$$\Phi_Q(\omega) = \frac{2k_BT}{\omega}\mathrm{Im}\alpha_Q(\omega) = 2k_BT\frac{R}{(L\omega^2 - \frac{1}{C})^2 + \omega^2 R^2}$$

$$\langle Q^2(t)\rangle = \int_{-\infty}^{\infty} \Phi_Q(\omega)\frac{d\omega}{2\pi} = \frac{2k_BTR}{L^2}\frac{1}{2\frac{R}{L}\frac{1}{LC}} = k_BTC$$

using the 1st hint. Equipartition means that the capacitor energy is $\frac{1}{2}\langle Q^2\rangle/C = \frac{1}{2}k_BT$, identical with the result above.

(c)

$$I = \dot{Q} \Rightarrow \langle I^2(t)\rangle = \int_{-\infty}^{\infty} \omega^2 \Phi_Q(\omega)\frac{d\omega}{2\pi} = \langle Q^2(t)\rangle\frac{1}{LC} = \frac{k_BT}{L}$$

where the 2nd hint is used. Equipartition means that the induction energy has an average $\frac{1}{2}L\langle I^2\rangle = \frac{1}{2}k_BT$, identical with the result above.

For the Nyquist result we need a finite interval of observed frequencies

$$\langle I^2\rangle_{\omega_1\leftrightarrow\omega_2} = 2k_BTR\cdot 2\int_{\omega_1}^{\omega_2} \frac{\omega^2}{(L\omega^2 - \frac{1}{C})^2 + \omega^2 R^2}\frac{d\omega}{2\pi}$$

where the factor 2 accounts for the negative frequency range. ω_1, ω_2 should be near the resonance

$$|L\omega_{1,2}^2 - \frac{1}{C}| \ll \omega_{1,2}R$$

and then the integral has just the range $\omega_1 - \omega_2$, leading to Nyquist's result.