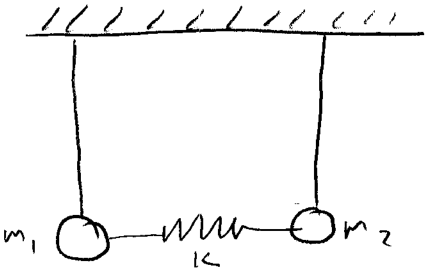
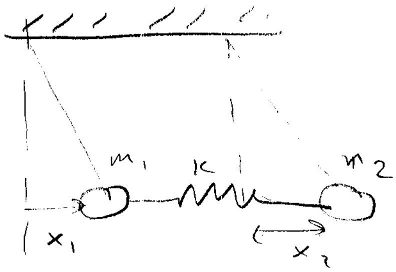
תדירויות עצמיות ואופני תנודה





$$\ddot{x}_1 = -\frac{g}{\ell}x_1 - \frac{k}{m_1}x_1 + \frac{k}{m_1}x_2$$

$$\ddot{x}_2 = -\frac{g}{\ell}x_2 - \frac{k}{m_2}x_2 + \frac{k}{m_2}x_1$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\underbrace{\begin{pmatrix} \frac{g}{\ell} + \frac{k}{m_1} & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{g}{\ell} + \frac{k}{m_2} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\ddot{\vec{X}} = -A\vec{X}$$

$$U^{-1}\ddot{\vec{X}} = -\underbrace{U^{-1}AU}_{D}\, U^{-1}\vec{X}$$

$$\frac{\partial^2}{\partial t^2}\left(U^{-1}\vec{X}\right) = -D\left(U^{-1}\vec{X}\right) \qquad \vec{y} \equiv U^{-1}\vec{X}$$

$$D \equiv \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

$$\frac{\partial^2}{\partial t^2}\vec{y} = -D\vec{y}$$

$$y_i = B_i \cos(\omega_i t + \varphi_i)$$

$$\det\left(A - \omega^2 \mathbb{1}\right) = 0$$

$$\begin{vmatrix} \frac{g}{\ell} + \frac{k}{m_1} - \omega^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{g}{\ell} + \frac{k}{m_2} - \omega^2 \end{vmatrix} = 0$$

$$\left(\frac{g}{\ell} + \frac{k}{m_1} - \omega^2\right)\left(\frac{g}{\ell} + \frac{k}{m_2} - \omega^2\right) - \frac{k^2}{m_1 m_2} = 0$$

$$\frac{g^2}{\ell^2} + \frac{gk}{\ell m_2} - \omega^2\frac{g}{\ell} + \frac{k}{m_1}\frac{g}{\ell} + \cancel{\frac{k^2}{m_1 m_2}} - \omega^2\frac{k}{m_1} - \omega^2\frac{g}{\ell} - \omega^2\frac{k}{m_2} + \omega^4 - \cancel{\frac{k^2}{m_1 m_2}}$$

$$\omega^4 - \omega^2\left(\frac{k}{m_1} + \frac{k}{m_2} + \frac{2g}{\ell}\right) + \frac{g}{\ell}\left(\frac{k}{m_1} + \frac{k}{m_2} + \frac{g}{\ell}\right) = 0$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

$$\omega^4 - \omega^2\left(\frac{K}{\mu} + \frac{2g}{\ell}\right) + \frac{g}{\ell}\left(\frac{K}{\mu} + \frac{g}{\ell}\right) = 0$$

$$\omega^2 = \frac{1}{2}\left[\frac{K}{\mu} + \frac{2g}{\ell} \pm \sqrt{\left(\frac{K}{\mu}\right)^2 + \cancel{4\frac{K}{\mu}\frac{g}{\ell}} + \cancel{4\left(\frac{g}{\ell}\right)^2} - \cancel{4\frac{gK}{\ell\mu}} - \cancel{4\frac{g^2}{\ell^2}}}\,\right] =$$

$$= \frac{1}{2}\left[\frac{K}{\mu} + \frac{2g}{\ell} \pm \frac{K}{\mu}\right]$$

$$\omega_1^2 = \frac{1}{2}\left(\frac{2g}{\ell} + \frac{2K}{\mu}\right) = \frac{g}{\ell} + \frac{K}{\mu}$$

$$\omega_2^2 = \frac{g}{\ell}$$

$$\underline{\omega_1^2 = \frac{g}{\ell} + \frac{K}{\mu}} \qquad \begin{pmatrix} -\frac{K}{m_2} & -\frac{K}{m_1} \\ -\frac{K}{m_2} & -\frac{K}{m_1} \end{pmatrix} \vec{V}_1 = 0 \qquad \vec{V}_1 = \frac{1}{\sqrt{m_1^2+m_2^2}}\begin{pmatrix} m_2 \\ -m_1 \end{pmatrix}$$

$$\omega_2^2 = \frac{g}{\ell} \qquad \begin{pmatrix} \frac{K}{m_1} & -\frac{K}{m_1} \\ -\frac{K}{m_2} & \frac{K}{m_2} \end{pmatrix} \vec{V}_2 = 0 \qquad \vec{V}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{m_2}{\sqrt{m_1^2+m_2^2}} & \frac{1}{\sqrt{2}} \\ -\frac{m_1}{\sqrt{m_1^2+m_2^2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \det(U) = \frac{m_2}{\sqrt{2}\sqrt{m_1^2+m_2^2}} + \frac{m_1}{\sqrt{2(m_1^2+m_2^2)}} =$$

$$=$$

$$U^{-1} = \frac{\sqrt{2(m_1^2+m_2^2)}}{m_1+m_2}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{m_1}{\sqrt{m_1^2+m_2^2}} & \frac{m_2}{\sqrt{m_1^2+m_2^2}} \end{pmatrix}$$

$$\vec{X} = U\vec{y} = A\vec{V}_1\cos(\omega_1 t + \varphi_1) + B\vec{V}_2\cos(\omega_2 t + \varphi_2)$$