Parallel capacitor

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The problem:

Consider a parallel capacitor made of two conductor disks of radius R being charged up by a constant electric current.

1. Find the electric and magnetic fields in the gap between the disks, as functions of the distance r from the axis and the time t (assume the charge is zero at $t = 0$).
2. Find the energy density and the Poynting vector in the gap. Note especially the direction of the vector.
3. Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap. Check that the power input is equal to the rate of increase of energy in the gap.

The solution:

1. The electric field in the gap is in the direction of $\hat{z}$.

$$V = \int \vec{E} \cdot d\vec{l} = Ed \tag{1}$$
$$Q = It \tag{2}$$
$$V = \frac{Q}{C} \tag{3}$$
$$C = \frac{A}{4\pi kd} = \frac{R^2}{4kd} \tag{4}$$

substituting (2),(3) and (4) into (1) we get

$$\vec{E} = \frac{4kIt}{R^2}\hat{z} \tag{5}$$

From the Maxwell's equation:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d}{dt}\left(\int \vec{E} \cdot d\vec{a}\right) \tag{6}$$

(where $\mu_0 I = 0$ because there is no current in the gap) we get

$$B_{(s)} 2\pi r = \mu_0 \varepsilon_0 \frac{d}{dt}\left(\frac{4kIt}{R^2}\pi r^2\right) \tag{7}$$

From the Ampere's law we conclude that the magnetic field has to be in the direction of $\hat{\theta}$ therefore

$$\vec{B} = \frac{2\mu_0 \varepsilon_0 kIr}{R^2}\hat{\theta} \tag{8}$$

2. The energy density is given by the formulae:

$$u_B = \frac{B^2}{2\mu_0} \tag{9}$$
$$u_E = \frac{\varepsilon_0 E^2}{2} \tag{10}$$

The Poynting vector is

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{1}{\mu_0} \frac{4kIt}{R^2} \hat{z} \times \frac{2\mu_0 \varepsilon_0 kIr}{R^2} \hat{\theta} = \frac{8\varepsilon_0 k^2 I^2 rt}{R^4} (-\hat{r}) \tag{11}$$

The minus sign means that the energy flows into the capacitor from the outside.

3. The total energy in the gap is

$$U = \iiint_V (u_B + u_E) dV = \int_0^{2\pi} \int_0^d \int_0^R \left(\frac{B^2}{2\mu_0} + \frac{\varepsilon_0 E^2}{2} \right) r d\theta dz ds = \tag{12}$$

$$= \int_0^{2\pi} \int_0^d \int_0^R \left(\frac{(2\mu_0 \varepsilon_0 kI)^2}{2R^4 \mu_0} r^3 + \frac{\varepsilon_0 (4kIt)^2}{2R4} r \right) d\theta dz ds = \varepsilon_0^2 k^2 d\pi I^2 + \frac{8\varepsilon_0 k^2 d\pi I^2 t^2}{R^2} \tag{13}$$

$$P_1 = VI = EdI = \frac{4kdI^2 t}{R^2} \tag{14}$$

$$P_2 = \frac{dU}{dt} = \frac{16\varepsilon_0 k^2 d\pi I^2 t}{R^2} ; \varepsilon_0 = \frac{1}{4\pi k} \rightarrow P_2 = \frac{4kdI^2 t}{R^2} \tag{15}$$

Notice that $P_1 = P_2$