Displecment Current

Submitted by: I.D. 043423755

The problem:
A long hollow cylinder is made out of a non-conducting material with a radius R, length l, and a charge surface density σ.
An external torque rotates the cylinder around its axis with a steady angular velocity of $\omega(t) = \alpha t$

1. Calculate the magnetic field inside the cylinder
2. What is the electrical field on the inner surface of the cylinder?
3. Calculate the Poynting vector on the inner surface of the cylinder.
4. Calculate the energy flux which enters the inner volume of the cylinder and deduce the conclusion that arises from this.

The solution:

The Ampere's law:

$$\oint B \cdot d\vec{l} = \mu_0 I \tag{1}$$

The current per a unit length is

$$I = \frac{Q/l}{T} = \frac{Q\omega}{2\pi l} \tag{2}$$

$$Q = \sigma(2\pi l R) \tag{3}$$

So the magnetic field here is like that of a coil

$$\vec{B} = \mu_0 I_0 n\hat{z} = \mu_0 \sigma R \omega \hat{z} = \mu_0 \sigma \alpha R t \hat{z} \tag{4}$$

2. Using the Faraday's law

$$\Phi = B \cdot A = \mu_0 \sigma \alpha R t \cdot \pi R^2 \tag{5}$$

Field $\vec{E}$ is in the tangential direction

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} \tag{6}$$

$$\frac{\Phi}{dt} = \mu_0 \sigma \alpha R^3 \pi \tag{7}$$

$$2\pi R E\hat{\theta} = -\mu_0 \sigma \alpha R^3 \pi \tag{8}$$

$$E_\theta = -\frac{1}{2}\mu_0 \sigma \alpha R^2 \tag{9}$$

3. The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{10}$$

$$= -\frac{1}{\mu_0}\frac{1}{2}\mu_0 \sigma \alpha R^2 \hat{\theta} \times \mu_0 \sigma R \alpha t \hat{z} = -\frac{1}{2}\mu_0 \sigma^2 \alpha^2 R^3 t \hat{r} \tag{11}$$

The meaning of the minus sign is that the vector pointing into the cylinder from outside, i.e. the energy flow is from outside.

4. The energy entering inside the cylinder is:

$$\begin{aligned}
\phi &= \int_s (-\vec{S}) \cdot d\vec{a} && (12)\\
&= \frac{1}{2}\mu_0 \sigma^2 \alpha^2 R^3 t \cdot 2\pi R l = \pi\mu_0(\sigma\alpha)^2 R^3 l \frac{1}{2}\frac{d(t)^2}{dt} && (13)\\
&= \frac{d}{dt}\left[\frac{1}{2}\pi\mu_0(\sigma\alpha)^2 R^4 l t^2\right] = \frac{d}{dt}\left(\frac{1}{2\mu_0}\left(\mu_0\sigma R\alpha t\right)^2 \cdot \left(\pi R^2 l\right)\right) && (14)\\
&= \frac{d}{dt}\left[\frac{1}{2\mu_0} BV\right] = P && (15)
\end{aligned}$$

where P is the rate of the magnetic enrgy produced inside the cylinder. The energy density of the electric field inside the cylinder is constant, so that it does not contribute to the power.
The power of the electric force opposing to the rotation is

$$\begin{aligned}
P_{\text{eff}} &= \vec{F} \cdot \vec{v} = -EQ\omega R && (16)\\
&= \left(\frac{1}{2}\mu_0 \sigma\alpha R^2\right)\left(\sigma 2\pi l R\right)\left(\alpha t R\right) && (17)\\
&= \frac{d}{dt}\left[\frac{1}{2\mu_0}\left((\mu 0)\sigma\alpha R t\right)^2\left(\pi R^2 l\right)\right] = \frac{d}{dt}\left(\frac{1}{2\mu_0} BV\right) && (18)
\end{aligned}$$

We obtained the the electric field is doing work for stopping the rotation. This work consumes all the electromagnetic field power entering from outside. Actually, this work of the electric field causes all the consumption of energy into the system.