Electromagnetic Induction

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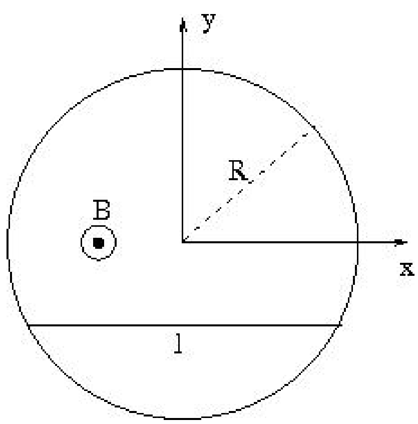
The problem:

Inside an infinite cylinder area $x^2 + y^2 \leq R^2$ there is a time-dependant magnetic field $\vec{B} = B_0(t)\hat{z}$ so that $\dot{B} = \text{const} > 0$. Outside the cylinder the magnetic field is zero. We place a conductor rod $l < 2R$ in $x - y$ plane so the tips of the rod touch the edge of the cylinder.

1. We close an electric circuit by connecting the two tips of the rod with straight conductors to the origin. What is the EMF in the circuit and the current direction in the rod?
2. Now we close an electric circuit by connecting the two tips of the rod with a conductor along the short arc. What is the EMF in the circuit and the current direction in the rod?
3. What is the potential between the two tips of the rod when they are not connected?
4. Explain if there is a contradiction between the results of the previous sections.



The solution:

The electmagnetic induction can be found from

$$\Phi_B = \oint_s B_0 \cdot ds = S \cdot B_0 \tag{1}$$

where the area S is constant. According to the Faraday's law

$$\varepsilon = -\dot{\Phi}_B = -S\dot{B}_0 \tag{2}$$

1. The area is a triangle, therefore,

$$S = \frac{l}{2}\sqrt{R^2 - \frac{l^2}{4}} \tag{3}$$

$$\varepsilon = -\frac{\dot{B}_0 l}{2}\sqrt{R^2 - \frac{l^2}{4}} \tag{4}$$

The current is in clockwise direction in the circuit - from right to left in the rod.

2. In order to calculate the new area we S' we calculate the area of the segment M and subtrack the area of the triangle from the preious section.

$$M = R^2 \arcsin \frac{l}{2R} \tag{5}$$

$$S' = M - S \tag{6}$$

$$\varepsilon = -\dot{B}_0 S' = -\dot{B}_0 \left(R^2 \arcsin \frac{l}{2R} - \frac{l}{4}\sqrt{R^2 - \frac{l^2}{4}} \right) \tag{7}$$

The current is in clockwise direction in the circuit - from left to right in the rod.

3. Inside the cylinder there is an electric field $\vec{E}$ according to:

$$\nabla \times \vec{E} = -\dot{B}_0 \hat{z} \tag{8}$$

$$\frac{1}{r}\left(\frac{\partial r \vec{E}_\varphi}{\partial r} - \frac{\partial \vec{E}_r}{\partial \varphi} \right) \hat{z} = -\dot{B}_0 \hat{z} \tag{9}$$

Because we do not have charges inside the cylinder so the radial field is zero.

$$\vec{E}_r = 0 \tag{10}$$

$$\frac{1}{r}\frac{\partial r \vec{E}_\varphi}{\partial r} \hat{z} = -\dot{B}_0 \hat{z} \tag{11}$$

$$\vec{E}_\varphi = -\frac{\dot{B}_0 r}{2} \tag{12}$$

$$\varepsilon = \int E_x dx = \int E_\varphi \cos(\varphi) dx \tag{13}$$

where x is a coordinate on the rod measured from its center, φ is the angle between E_φ and the rod and also the angle between $r\hat{r}$ and the perpendicular from the center of the circle to the chord.

$$\cos(\varphi) = \frac{E_x}{E_\varphi} = \frac{|y|}{r} \tag{14}$$

$$|y| = \sqrt{R^2 - \frac{l^2}{4}} \tag{15}$$

$$\varepsilon = \int \vec{E}_\varphi \frac{|y|}{r} dx \tag{16}$$

$$\varepsilon = -\frac{\dot{B}_0}{2} \int_{l/2}^{-l/2} \sqrt{R^2 - \frac{l^2}{4}} \vec{d}x \tag{17}$$

$$\varepsilon = -\dot{B}_0 \frac{l}{2} \sqrt{R^2 - \frac{l^2}{4}} \tag{18}$$

4. We can see that result of the the first and the third sections are equal. The reason is that because if we calculate the EMF in 1 like the calculation in 3 but we take the entire loop of the circuit:

$$\oint_L \vec{E}_\varphi \vec{d}l = \varepsilon \tag{19}$$

where L is the the perimeter of triangle. The integral over this loop is not zero only on the rod l, because there is no radial component of the field, so that the integrals over the radial sides of the triangle are zero. However, in the section 2 the integral over the arc is not zero.