Capacitors discharging

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The problem:

A spherical conductor with a radius R_1 is in a conducting spherical shell with a radius R_2. The sphere and the shell are concentric.
The two conductors are connected to a constant electric source V.

1. What is the charge of the sphere and the shell?
2. What is the electric field?

Now the electric source is disconnected, and the space between the sphere and the shell is filled with a conducting material σ .

1. How much time takes for the charge on the sphere to lower to a half of its initial value ?
2. How much energy turns into heat during the process ?

The solution:

Assuming the potential difference V between the conductors we know that the capacitance of the system is

$$C = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1} \tag{1}$$

and then

$$Q = CV = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1} V \tag{2}$$

The electric field is

$$\vec{E} = \begin{cases} 0, & r < R_1 \\ k\frac{Q}{r^2}\hat{r}, & R_1 < r < R_2 \\ 0, & r > R_2 \end{cases} \tag{3}$$

After disconnecting the external voltage, and filling the space between the conductors with conducting material, the capacitor is being discharged through it, and the its charge is

$$Q(t) = Q_0 \mathrm{e}^{-t/\tau}, \quad \tau = RC \tag{4}$$

We know C, but what is R?

First way
Suppose that there is a potential V. It gives rise to a radial current I, and then $R = V/I$ by the Ohm's law.

$$V = -\int_{R_1}^{R_2} \vec{E} \cdot d\vec{r} \tag{5}$$

$$\vec{E} = \frac{1}{\sigma}\vec{J} \tag{6}$$

Then remembering that $J = I/A$ for homogeneous current we get

$$V = \frac{1}{\sigma}\int_{R_1}^{R_2} J dr \tag{7}$$

$$= \frac{1}{\sigma}\int_{R_1}^{R_2} \frac{I}{4\pi r^2} dr = \frac{I}{4\pi\sigma}\frac{R_2 - R_1}{R_1 R_2} \tag{8}$$

Therefore,

$$R = \frac{V}{I} = \frac{1}{4\pi\sigma}\frac{R_1 - R_2}{R_1 R_2} \tag{9}$$

Second way
For a wire of a length L, area A and conductivity σ the resistance is $R = \frac{L}{\sigma A}$.
Since the current in our case is radial, we have many infinitecimal thin spheres in series.

$$dR = \frac{dr}{\sigma A} = \frac{dr}{\sigma 4\pi r^2} \tag{10}$$

$$R = \int dR = \int_{R_1}^{R_2} \frac{dr}{\sigma 4\pi r^2} = \frac{1}{4\pi\sigma}\frac{R_1 - R_2}{R_1 R_2} \tag{11}$$

Going back to our problem

$$\tau = RC = \frac{\epsilon_0}{\sigma} \tag{12}$$

To know when the charge lowers to the half of its initial value we have to solve

$$Q_0 \mathrm{e}^{-t_{1/2}/\tau} = \frac{Q_0}{2} \tag{13}$$

$$t_{1/2} = \tau \ln 2 = \frac{\epsilon_0}{\sigma} \ln 2 \tag{14}$$

For calculating the heat we use

$$dU = V dQ \tag{15}$$

For the resistor

$$dU = V dQ = IR \cdot I dt = I^2 R dt \tag{16}$$

$$I = \frac{dQ}{dt} = -\frac{Q_0}{\tau}\mathrm{e}^{-t/\tau} \tag{17}$$

$$U = \int dU = \int_0^{t_{1/2}} I^2 R dt = \frac{3}{4}\frac{Q_0^2}{2C} \tag{18}$$

For the capacitor

$$dU = V dQ = \frac{Q}{C} dQ \tag{19}$$

$$U = \int dU = \int_{Q_0}^{Q_0/2} \frac{Q}{C} dQ = -\frac{3}{4}\frac{Q_0^2}{2C} \tag{20}$$

We obtained the same energies (up to the sign), and this is because the energy of the capacitor decreases and turns into heat over the resistor.