Electric dipole and capacitors

The problem:

A cylindrical capacitor has length L. The capacitor consists of two conducting cylinderical shells with a common axis and radii b_1, b_2 ($b_2 > b_1$).

1. What is the capacity of the capacitor?
2. Find the force that acts on the inner cylinder, if it is being pulled along the common axis up to ΔL above the upper part of the capacitor. ($\Delta L \ll L$)

The solution:

1. The capacity is defined as $C = \frac{Q}{V}$. First, we will calculate the field between the cylinderical shells using the Gauss's law

$$E 2\pi r l = 4\pi k \frac{Q}{L} l \tag{1}$$

$$E = \frac{2kQ}{Lr} \tag{2}$$

$$\Delta\phi = -\int_{b_1}^{b_2} \frac{2kQ}{Lr} dr = -\frac{2kQ}{L} \ln\frac{b_2}{b_1} \tag{3}$$

$$C = \frac{Q}{V} = \frac{L}{2k \ln\frac{b_2}{b_1}} = 2\pi\epsilon_0 \frac{L}{\ln\frac{b_2}{b_1}} \tag{4}$$

2. Since ΔL is very small, the change of the capacity will also be very small. The energy of a capacitor is given by: $U = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$. So we presume that the change in the energy will be very small.

$$C - \Delta C = 2\pi\epsilon_0 \frac{L - \Delta L}{\ln\frac{b_2}{b_1}} \tag{5}$$

$$\Delta U = U(C) - U(C - \Delta(C)) = \frac{Q^2}{2}\left(\frac{1}{C} - \frac{1}{C - \Delta C}\right) \approx -\frac{Q^2}{2}\frac{\Delta C}{C^2} = -\frac{1}{2}\frac{Q^2 \Delta L}{LC} \tag{6}$$

$$F = -\frac{\Delta U}{\Delta L} = \frac{1}{2}\frac{Q^2}{LC} = \frac{Q^2}{4\pi\epsilon_0 L^2} \ln\frac{b_2}{b_1} \tag{7}$$

The direction of the force is downwards.