Conducting sphere inside electric field

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The problem:

A neutral conducting sphere with a radius R is placed inside a constant field $\vec{E} = E_0\hat{z}$

1. Find the dipole induced by the sphere.
2. What is the electric field outside the sphere?
3. What is the charge distribution on the sphere shell?

The solution:

1. As we know the electric field inside a conducting sphere is 0.

The electric field induced inside a polarized sphere is

$$\vec{E}_d = -k\frac{\vec{p}}{R^3} \tag{1}$$

Therefore:

$$\begin{aligned} \vec{E}_z - \vec{E}_d &= 0 & (2)\\ E_0\hat{z} - \frac{k\vec{p}}{R^3} &= 0 & (3)\\ & & (4) \end{aligned}$$

and

$$\vec{p} = \frac{R^3 E_0}{k}\hat{z} \tag{5}$$

The electric field outside the sphere is just the sum of the field induced by the sphere and the constant external field:

$$\begin{aligned} \vec{E} &= k\frac{3(\vec{p}\cdot\vec{r})\vec{r} - r^2\vec{p}}{r^5} + E_0\hat{z} & (6)\\ &= k\frac{3p_z z\vec{r} - r^2 p_z\hat{z}}{r^5} + E_0\hat{z} = E_0\left(R^3\frac{3z\vec{r} - r^2\hat{z}}{r^5} + \hat{z}\right) & (7) \end{aligned}$$

Outside of the sphere the normal component of the electric field to the surface is

$$\begin{aligned} E_r &= \vec{E}\cdot\hat{r} = E_0\left(R^3\frac{3z\vec{r} - r^2\hat{z}}{r^5} + \hat{z}\right)\hat{r} = E_0\left(R^3\frac{3z - r\cos\theta}{r^4} + \cos\theta\right) & (8)\\ &= E_0\left(R^3\frac{2\cos\theta}{r^3} + \cos\theta\right) & (9) \end{aligned}$$

where θ is the angle between $\vec{r}$ and $\hat{z}$. Then at $r = R$

$$E_r^+ = 3E_0\cos\theta \tag{10}$$

whereas inside the sphere the field is zero.
By the Gauss' law

$$\vec{E}_r^+ - \vec{E}_r^- = 4\pi k\sigma \tag{11}$$

Therefore,

$$\sigma = \frac{3E_0\cos\theta}{4\pi k} \tag{12}$$