Electric field

The problem:

Two identical particles, with the charge $+q$, are held in place at a distance d from each other. A particle charged $-Q$ with the mass m is placed between the two, in the middle. The particle $(-Q)$ is slightly diverted upwards, perpendicular to the straight line connecting the two particles $(+q)$, and let go. Show that harmonic oscillations occur, so that:

$$T^2 = \frac{\varepsilon_0 m \pi^3 d^3}{Qq} \tag{1}$$

The solution:

We define the axis on which the positive charges lay as x. y axis is perpendicular to x, positive direction set upwards. The origin is set between the charges. The negative charge $-Q$ can move along the y axis.
The Coulomb's Law :

$$\Sigma \vec{F} = \frac{kqq_i(\vec{r} - \vec{r}_i)}{\|\vec{r} - \vec{r}_i\|^3} \tag{2}$$

$\vec{r}_1$ and $\vec{r}_2$ are position vectors of positively charged particles and $\vec{r}$ is the position of the moving charge.

$$\Sigma \vec{F} = \frac{kq(-Q)(\vec{r} - \vec{r}_1)}{\|\vec{r}_1\|^3} + \frac{kq(-Q)(\vec{r} - \vec{r}_2)}{\|\vec{r}_2\|^3} \tag{3}$$

where

$$\vec{r} = \Delta y \hat{y} \tag{4}$$

$$\vec{r}_1 = \left(\frac{d}{2}\right)\hat{x} \tag{5}$$

$$\vec{r}_2 = \left(-\frac{d}{2}\right)\hat{x} \tag{6}$$

Therefore,

$$\Sigma \vec{F} = \frac{kQq[\left(\frac{d}{2}\right)\hat{x} - (\Delta y)\hat{y}]}{[\left(\frac{d}{2}\right)^2 + \Delta y^2]^{\frac{3}{2}}} + \frac{kQq[\left(-\frac{d}{2}\right)\hat{x} - (\Delta y)\hat{y}]}{[\left(\frac{d}{2}\right)^2 + \Delta y^2]^{\frac{3}{2}}} = \frac{-2kQq\Delta y}{[\left(\frac{d}{2}\right)^2 + \Delta y^2]^{\frac{3}{2}}}\hat{y} \tag{7}$$

simplifying, and substituting k with ε_0 ($k \equiv \frac{1}{4\pi\varepsilon_0}$) we get :

$$\Sigma \vec{F} = \frac{-4Qq\Delta y}{\pi\varepsilon_0 d^3[1 + \left(\frac{2\Delta y}{d}\right)^2]^{\frac{3}{2}}}\hat{y} \tag{8}$$

Extracting taylor series to first order, using the relation:

$$\frac{1}{(1+\mathrm{x})^{\frac{3}{2}}} = 1 - \frac{3}{2}x + \mathrm{O}(x^2) \tag{9}$$

We get :

$$\Sigma \vec{F} = \frac{-4Qq\Delta y}{\pi\varepsilon_0 d^3}\hat{y} \quad \text{or} \quad m\ddot{y} = \frac{-4Qq\Delta y}{\pi\varepsilon_0 d^3} \tag{10}$$

$$\Rightarrow \omega^2 = \frac{4Qq}{m\pi\varepsilon_0 d^3} \quad \text{where} \quad T^2 = \frac{4\pi^2}{\omega^2} \tag{11}$$

Finally

$$T^2 = \frac{\varepsilon_0 m\pi^3 d^3}{Qq} \tag{12}$$