The electric field

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The problem:

An isolated electrical wire is charged uniformly with charge q and bent into a circular shape (radius R) with a small hole $b \ll R$ (where b is the arc length).
What is the electrical field in the middle of the circle?

The solution:

The simple solution is to use superposition.
The electric field in the middle of a complete ring is, of course, zero.
Now we'll sum up the field of a complete ring with the field of a very small wire of the same size and shape as the hole but with a negative charge.
Because $b \ll R$ the wire can be taken as a *negative* point charge and, therefore, the field is $\vec{E} = \frac{k\lambda b}{R^2}\hat{x}$ (when we take the hole to be on the X axis and $\lambda = \frac{q}{2\pi R}$).

It is also possible to calculate the field directly. Taking $\vec{r} = (0,0,0)$ and $\vec{r}' = (R\cos\theta, R\sin\theta, 0)$ we have

$$d\vec{E} = \frac{kdq}{R^3}(-R\cos\theta, -R\sin\theta, 0) \tag{1}$$

$$dq = \lambda R d\theta \tag{2}$$

$$\vec{E} = \int_{\alpha}^{-\alpha} \frac{k\lambda R d\theta'}{R^3}(-R\cos\theta', -R\sin\theta', 0) = \frac{k\lambda}{R}(2\sin\alpha, 0, 0) \tag{3}$$

where $\pm\alpha$ are the angles of the edges of the bent wire (or, the upper and lower limits of the hole).
Since $b \ll R$ we can approximate $\tan\alpha \simeq \sin\alpha = \frac{b/2}{R}$.
Substituting into the expression for the field we obtain

$$\vec{E} = \frac{k\lambda b}{R^2}\hat{x} \tag{4}$$