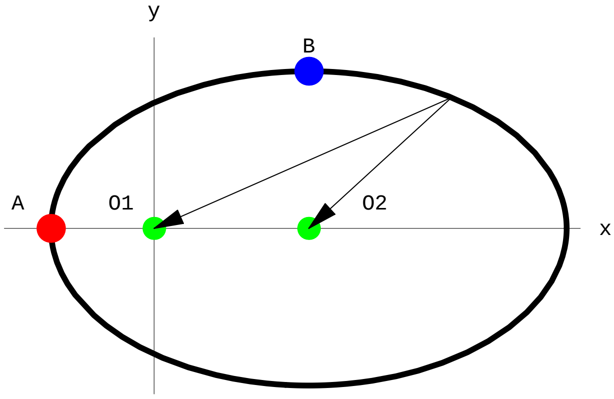
radius R in the plane $x-y$. The light frequency in the source frame is f. A distant observer (in the same $x-y$ plane) measures the frequency of received light as a function of time. Find this function.

10 Problems given at various tests, including midterms and finals

Problem 10.1. A particle of the mass m is moving on the elliptic orbit $r = p/(1-\epsilon\cos\varphi)$. The particle is attracted to the focus O_1 by the force $|\boldsymbol{F}_1| = k_1/r_1^2$ (where $|\boldsymbol{r}_1|$ is the distance between the focus O_1 and the particle). In addition, it is attracted to the ellipse center O_2 with the force $|\boldsymbol{F}_2| = k_2 r_2$. The third force acting on the particle is always perpendicular to the particle velocity. When the particle is in A, its velocity is $\boldsymbol{v}_1 = (0, v_1)$. What is its angular momentum in B ?



Solution. Points:

First we need the coordinates of the points A, B, O_1, and O_2:

1. O_1 is the coordinate origin, so that $r_{O_1} = x_{O_1} = y_{O_1} = 0$.

2. A corresponds to $\varphi_A = \pi$, so that $r_A = p/(1+\epsilon)$, $x_A = -p/(1+\epsilon)$, $y_A = 0$.

3. It is convenient to use the point C: $\varphi_C = 0$, $r_C = p/(1-\epsilon)$, $x_C = p/(1-\epsilon)$, $y_C = 0$.

4. O_2 is in the middle between A and C, so that $x_{O_2} = (x_A + x_C)/2 = \epsilon p/(1-\epsilon^2)$, $y_{O_2} = (y_A + y_C)/2 = 0$.

5. B is just above O_2, so that $x_B = x_{O_2} = \epsilon p/(1-\epsilon^2)$. The equation $r = p(1-\epsilon\cos\varphi)$ can be written as

$$r - \epsilon r\cos\varphi = p \rightarrow r = p + \epsilon x, \quad x = r\cos\varphi$$

so that $r_B = p/(1-\epsilon^2)$. Now

$$y_B = \sqrt{r_B^2 - x_B^2} = \frac{p}{\sqrt{1-\epsilon^2}}$$

Force vectors:

We denote the position of the particle with $\boldsymbol{r} = (x, y)$, respectively, other points $\boldsymbol{r}_{O_1} = (0,0)$, $\boldsymbol{r}_{O_2} = (x_{O_2}, 0)$, $\boldsymbol{r}_A = (x_A, 0)$, $\boldsymbol{r}_B = (x_B, y_B)$.

Now we introduce the vectors $\boldsymbol{r}_1 = \boldsymbol{r} - \boldsymbol{r}_{O_1} = \boldsymbol{r}$ and $\boldsymbol{r}_2 = \boldsymbol{r} - \boldsymbol{r}_{O_2}$. With this notation the forces (both *attractive*) will be written as follows

$$\boldsymbol{F}_1 = -\frac{k_1}{r_1^2}\widehat{\boldsymbol{r}}_1,$$
$$\boldsymbol{F}_2 = -k_2 r_2 \widehat{\boldsymbol{r}}_2$$

where $\widehat{\boldsymbol{r}}_1$ and $\widehat{\boldsymbol{r}}_2$ are unit vectors, $r_1 = |\boldsymbol{r}_1|$ and $r_2 = |\boldsymbol{r}_2|$.

Potential energy:

Both forces are central forces: the direction of $\boldsymbol{F}_1$ is to O_1 and the magnitude depends only on $|mbr_1|$, the direction of $\boldsymbol{F}_2$ is to O_2 and the magnitude depends only on $|mbr_2|$. Central forces are conservative, and the relation to the potential energy is the following:

$$\boldsymbol{F}_1 = -\frac{\partial U_1}{\partial r_1}\widehat{\boldsymbol{r}}_1, \qquad \boldsymbol{F}_2 = -\frac{\partial U_2}{\partial r_2}\widehat{\boldsymbol{r}}_2$$

which gives

$$\frac{\partial U_1}{\partial r_1} = \frac{k_1}{r_1^2}, \qquad \frac{\partial U_2}{\partial r_2} = k_2 r_2$$

and finally

$$U_1 = -\frac{k_1}{r_1}, \qquad U_2 = \frac{k_2 r_2^2}{2}$$

Energy conservation:

Since the third (unknown) force is always perpendicular to the velocity, $\boldsymbol{F}_3 \perp \boldsymbol{v}$, it does not produce work, so that the energy conservation gives

$$\frac{m\boldsymbol{v}^2}{2} + U_1 + U_2 = \frac{m\boldsymbol{v}^2}{2} - \frac{k_1}{r_1} + \frac{k_2 r_2^2}{2} = E = \text{const}$$

When the particle is in the point A, we have $\boldsymbol{v} = (0, v_1)$, $r_1 = p/(1+\epsilon)$, $r_2 = p/(1-\epsilon^2)$.

When the particle is in the point B, we have $\boldsymbol{v} = (0, v)$ (this v is the unknown we are looking for), $r_1 = p/(1-\epsilon^2)$, $r_2 = p/\sqrt{1-\epsilon^2}$.

Therefore, we have

$$\frac{mv_1^2}{2} - \frac{k_1(1+\epsilon)}{p} + \frac{k_2 p^2}{2(1-\epsilon^2)^2} = \frac{mv^2}{2} - \frac{k_1(1-\epsilon^2)}{p} + \frac{k_2 p^2}{2(1-\epsilon^2)}$$

Finally

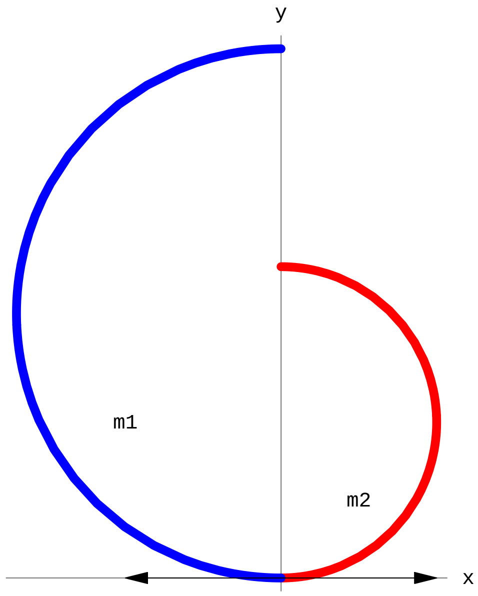
$$v = \left[v_1^2 - \frac{2k_1\epsilon(1-\epsilon)}{mp} + \frac{k_2\epsilon^2 p^2}{(1-\epsilon^2)^2}\right]^{1/2}$$

Angular momentum:

$\boldsymbol{J} = m\boldsymbol{r} \times \boldsymbol{v}$. In the point B the velocity is $\boldsymbol{v} = (v, 0, 0)$, so that

$$J_z = -mvy_B = -\frac{mvp}{\sqrt{1-\epsilon^2}}$$

Problem 10.2. A particle, initially resting in the coordinate origin, suddenly breaks up into three particles with the masses m_1, m_2, and m_3. The particle m_1 has the charge $q > 0$. It starts moving into negative x-direction in the homogeneous magnetic field $\boldsymbol{B} = (0, 0, B)$. After having completed half a circle the particle finds itself at the distance l_1 from the starting point. The particle m_2 has the charge $-q < 0$. It starts moving in the positive x-direction, and after having completed half a circle is at the distance l_2 from the starting point. What is the velocity of the third particle ? The magnetic force is $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$.



Solution. For each particle in the magnetic field $m\boldsymbol{a} = q\boldsymbol{v} \times \boldsymbol{B}$. Since $\boldsymbol{v} \perp \boldsymbol{B}$, we can write $m|\boldsymbol{a}| = |q||\boldsymbol{v}||\boldsymbol{B}|$, or

$$\frac{mv^2}{r} = |q|B$$

which gives

$$v = \frac{|q|Br}{m}$$

From this expression we immediately find ($l = 2r$ for a semicircle !)

$$v_1 = \frac{qBl_1}{2m_1}, \qquad v_2 = \frac{qBl_2}{2m_2}$$

Momentum conservation gives

$$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v_2} + m_3\boldsymbol{v}_3 = 0$$

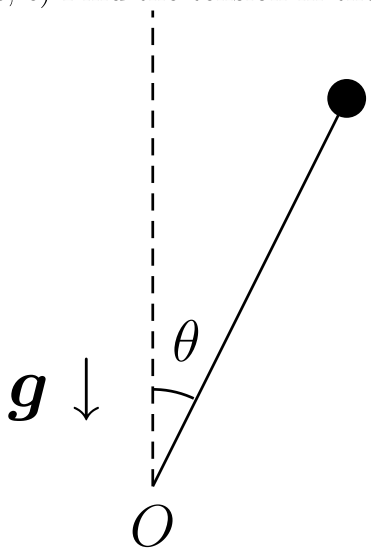
or

$$\boldsymbol{v}_3 = -\frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_3}$$

Substituting $\boldsymbol{v}_1 = v_1\hat{\boldsymbol{e}_x}$, $\boldsymbol{v}_2 = -v_2\hat{\boldsymbol{e}_x}$, we find

$$\boldsymbol{v}_3 = \frac{m_2v_2 - m_1v_1}{m_3}\hat{\boldsymbol{e}_x} = \frac{qB(l_2 - l_1)}{2m_3}\hat{\boldsymbol{e}_x}$$

Problem 10.3. A particle of the mass m is connected with the point O with the use of a massless rod of the length L. The rod can freely rotate around O. When the angle between the rod and the vertical is θ_0 the particle velocity is $\boldsymbol{v}_0 = v_1\hat{\boldsymbol{\theta}} + v_2\hat{\boldsymbol{\varphi}}$. a) What is conserved ? b) Find v_θ and v_φ (or $\dot{\theta}$ and $\dot{\varphi}$) as functions of θ; c) Find $\ddot{\theta}$ and $\ddot{\varphi}$ as functions of θ; d) Find the maximum speed $|\boldsymbol{v}|_{max}$ of the particle; e) Find the tension in the rod as a function of θ.



Solution. Preparations:

$$\begin{aligned}\boldsymbol{F} &= -mg\hat{\boldsymbol{z}} + T\hat{\boldsymbol{r}}\\ \boldsymbol{v} &= v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}}\\ &= r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\varphi}\hat{\boldsymbol{\varphi}}\\ \dot{r} &= 0\end{aligned}$$

a):

$$\begin{aligned}E &= \frac{mv^2}{2} + U, \qquad U = mgz = mgr\cos\theta\\ E &= \frac{m}{2}(v_\theta^2 + v_\varphi^2) + mgr\cos\theta\\ &= \frac{m}{2}(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) + mgr\cos\theta\end{aligned}$$

$$
\begin{aligned}
&\dot{E} = \boldsymbol{T}\cdot\boldsymbol{v} = T\hat{\boldsymbol{r}}\cdot(v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}}) = 0\\
&E = \text{const} = \frac{m}{2}(v_1^2 + v_2^2) + mgL\cos\theta_0\\
&\dot{\boldsymbol{p}} = \boldsymbol{F} \neq 0\\
&\boldsymbol{J} = \boldsymbol{r}\times m\boldsymbol{v} = mr\hat{\boldsymbol{r}}\times(v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}})\\
&\quad = mrv_\theta\hat{\boldsymbol{\varphi}} - mrv_\varphi\hat{\boldsymbol{\theta}}\\
&\dot{\boldsymbol{J}} = \boldsymbol{r}\times\boldsymbol{F} = -rmg\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}\\
&\dot{J}_z = 0\\
&J_z = \boldsymbol{J}\cdot\hat{\boldsymbol{z}} = mr\sin\theta v_\varphi = mr^2\sin^2\theta\dot{\varphi}\\
&J_z = \text{const} = mv_2L\sin\theta_0
\end{aligned}
$$

b):

$$
\begin{aligned}
v_\varphi &= \frac{J_z}{mr\sin\theta}\\
\dot{\varphi} &= \frac{v_\varphi}{r\sin\theta} = \frac{J_z}{mr^2\sin^2\theta}\\
\frac{m}{2}&(v_\theta^2 + v_\varphi^2) + mgr\cos\theta = E\\
v_\theta &= \pm\sqrt{\frac{2E}{m} - v_\varphi^2 - 2gr\cos\theta}\\
&= \pm\sqrt{\frac{2E}{m} - \frac{J_z^2}{m^2r^2\sin^2\theta} - 2gr\cos\theta}\\
\dot{\theta} &= \frac{v_\theta}{r} = \pm\frac{1}{r}\sqrt{\frac{2E}{m} - \frac{J_z^2}{m^2r^2\sin^2\theta} - 2gr\cos\theta}
\end{aligned}
$$

c):

$$
\begin{aligned}
\ddot{\theta} &= \frac{d\dot{\theta}}{d\theta}\cdot\dot{\theta}\\
&= \frac{d}{d\theta}\left(\pm\frac{1}{r}\sqrt{\frac{2E}{m} - \frac{J_z^2}{m^2r^2\sin^2\theta} - 2gr\cos\theta}\right)\\
&\cdot\left(\pm\frac{1}{r}\sqrt{\frac{2E}{m} - \frac{J_z^2}{m^2r^2\sin^2\theta} - 2gr\cos\theta}\right)\\
&= \pm\frac{1}{r}\left(\frac{2E}{m} - \frac{J_z^2}{m^2r^2\sin^2\theta} - 2gr\cos\theta\right)^{-1/2}\\
&\cdot\left(\frac{J_z^2\cos\theta}{m^2r^2\sin^3\theta} + gr\sin\theta\right)\dot{\theta}\\
&= \frac{1}{r^2}\left(\frac{J_z^2\cos\theta}{m^2r^2\sin^3\theta} + gr\sin\theta\right)
\end{aligned}
$$

$$\begin{aligned}\ddot{\varphi} &= \frac{d\dot{\varphi}}{d\theta} \cdot \dot{\theta} \\ &= \mp \frac{1}{r} \sqrt{\frac{2E}{m} - \frac{J_z^2}{m^2 r^2 \sin^2 \theta} - 2gr \cos \theta} \left(\frac{2J_z \cos \theta}{mr^2 \sin^3 \theta} \right)\end{aligned}$$

d):

$$\begin{aligned}&v^2 = \frac{2}{m}(E - mgr \cos \theta) \\ &\frac{d}{d\theta} v^2 = 2gr \sin \theta > 0 \\ &v = v_{max} \to \theta = \theta_{max} > \pi/2 \\ &\theta = \theta_{max} \Rightarrow \dot{\theta} = 0 \Rightarrow \\ &\frac{2E}{m} - \frac{J_z^2}{m^2 r^2 \sin^2 \theta_{max}} - 2gr \cos \theta_{max} = 0\end{aligned}$$

e):

$$\begin{aligned}&(\boldsymbol{T} + m\boldsymbol{g}) \cdot \hat{\boldsymbol{r}} = -\frac{mv^2}{r} \\ &T - mg \cos \theta = -\frac{mv^2}{r} = -\frac{2E - 2mgr \cos \theta}{r} \\ &T = 3mg \cos \theta - \frac{2E}{r}\end{aligned}$$

Problem 10.4. A particle moves according to $r = r_0 \exp(\varphi)$ in the central force. Find $U(r)$.

Solution. In the central force energy and angular momentum are conserved. Energy conservation gives

$$E = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\varphi}^2 + U = \text{const}.$$

Angular momentum conservation gives:

$$mr^2 \dot{\varphi} = J = \text{const}.$$

and therefore

$$\dot{\varphi} = \frac{J}{mr^2}.$$

From the given relation $r = r_0 \exp(\varphi)$ we have

$$\dot{r} = (d/dt)(r_0 \exp(\varphi)) = r_0 \exp(\varphi) \dot{\varphi} = r \dot{\varphi} = \frac{J}{mr}.$$

Substituting (10.4) and (10.4) into the energy conservation (10.4) we finally obtain

$$E = \frac{J^2}{mr^2} + U$$

and

$$U = E - \frac{J^2}{mr^2}$$

Problem 10.5. A particle moves according to

$$x = A\exp(-\gamma t)\cos(\omega t)$$
$$y = A\exp(-\gamma t)\sin(\omega t)$$

Find tangential acceleration.

Solution. The problem is solved by direct differentiating: $v_x = \dot{x}$, $v_y = \dot{y}$, $a_x = \dot{v}_x$, $a_y = \dot{v}_y$, so that the velocity and acceleration are:

$$\begin{aligned} v_x &= \dot{x} = -\gamma A\exp(-\gamma t)\cos(\omega t) - \omega A\exp(-\gamma t)\sin(\omega t) = -\gamma x - \omega y, \\ v_y &= -\gamma y + \omega x, \\ a_x &= -\gamma v_x - \omega v_y = (\gamma^2 - \omega^2)x + 2\gamma\omega y, \\ a_y &= -\gamma v_y + \omega v_x = (\gamma^2 - \omega^2)y - 2\gamma\omega x. \end{aligned}$$

Tangential acceleration is parallel to the velocity vector, so that $a_\parallel = \boldsymbol{a}\cdot\boldsymbol{v}/|\boldsymbol{v}|$:

$$\begin{aligned} a_\parallel &= (a_x v_x + a_y v_y)/(v_x^2 + v_y^2)^{1/2} \\ &= -\gamma\sqrt{\gamma^2 + \omega^2}A\exp(-\gamma t) \end{aligned}$$

Problem 10.6. A particle of the mass m moves in the potential energy $U(x) = -Ax^2/2 + Bx^3/3$, $A > 0$, $B > 0$. Find the frequency of small oscillations.

Solution. Equilibrium is where $F_x = -(dU/dx) = 0$:

$$-\frac{dU}{dx} = -Ax + Bx^2 = 0 \to x_1 = 0, x_2 = A/B.$$

Whether the equilibrium is stable or not is determined by the second derivative $(d^2U/dx^2) = -A + 2Bx$. If $(d^2U/dx^2) < 0$ the point is a maximum and the equilibrium is unstable, if $(d^2U/dx^2) > 0$ the point is a minimum and the equilibrium is stable. In $x = x_1 = 0$ we have $(d^2U/dx^2) = -A < 0$ so that this is an unstable equilibrium. In $x = x_2 = A/B$ we have $(d^2U/dx^2) = A > 0$ and this is a stable equilibrium.

Near the equilibrium point the potential energy can be Tayor expanded:

$$U(x) \approx U(x_2) + \tfrac{1}{2}A(x - x_2)^2.$$

Let us denote $X = x - x_2$ then $v_x = \dot{x} = \dot{X}$ and the energy conservation is written as

$$\tfrac{1}{2}m\dot{X}^2 + \tfrac{1}{2}AX^2 = \text{const}$$

which immediately gives (according to the general rule)

$$\omega^2 = A/m.$$

Problem 10.7. A hollow cylinder is sliding without friction (no rolling) with the velocity v. The cylinder comes to a surface with friction. What is the final velocity of the cylinder ?

Solution. When the cylinder comes to the surface with friction it is decelerated by the friction force and at the same time its rotation is acceletrated until the cylinder begins to roll without sliding. In the rolling state the friction force is zero and the velocity does not change. Let the friction force magnitude be F_s. Then the deceleration is

$$m\dot{V} = -F_s \to V(t) = v - \frac{F_s}{m}t,$$

where$V(t)$ is the velocity of the center-of-mass in the moment t. The moment of the friction force (torque) accelerates the rotation around the axis passing through the center-of-mass, as follows:

$$I\dot{\omega} = F_s r \to \omega(t) = \frac{F_s r}{I}t,$$

where $\omega(t)$ is the angular velocity of rotation and I is the moment of inertia around the axis passing through the center-of-mass. According to the definition $I = \sum m_i r_i^2$, where r_i is the distance from the rotating mass (small part of the body) from the rotation axis. All points of the cylinder are at the same distance r from the rotation axis, so that we have

$$I = \sum m_i r_i^2 = \sum_i m_i r^2 = (\sum_i m_i) r^2 = mr^2.$$

From (10.7) and (10.7) we find

$$\omega(t) = \frac{F_s}{mr}t.$$

The velocity and angular velocity stop changing when sliding stops. The condition of the rolling without sliding is $V = \omega r$ which gives

$$v - \frac{F_s}{m}t = \frac{F_s}{mr}tr = \frac{F_s}{m}t,$$

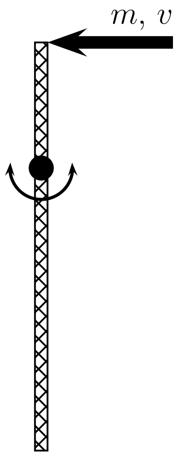
and therefore, in the moment when the velocity stops changing

$$\frac{F_s}{mr}t = v/2.$$

Substituting into (10.7) we have

$$V_{final} = v/2.$$

Problem 10.8. A rod of the mass M and length l can rotate in the vertical plane around the axis which passes through the point at $a < l/2$ from its upper end. A bullet of the mass $m \ll M$ with the horizontally directed velocity v strikes the rod at the upper end and remains in it. What is



the rotation angle of the rod ?

Solution. Angular momentum is conserved during the collision, which means

$$I\omega = mva \rightarrow \omega = mva/I,$$

where ω is the angular velocity of the rotation just after the collision, and I is the moment of inertia. After the collision the energy is conserved, that is, the initial kinetic energy goes into the potential energy as the bar rotates to the angle θ and its center-of-mass rises. Therefore, for the maximum angle we have

$$I\omega^2/2 = MgL(1 - \cos\theta) \rightarrow \cos\theta = 1 - I\omega^2/2MgL.$$

where $L = l/2 - a$ is the distance between the axis and center-of-mass of the bar (we neglect the mass of the bullet).

Calculation of I: Let us choose coordinate origin in the axis, and the positive direction downwards.

Then

$$I = \int_{-a}^{l-a} x^2(M/l)dx = \frac{M}{3l}[(l-a)^3 + a^3]$$

Combine all calculations.

Problem 10.9. An electron moving with the energy $E \gg mc^2$ toward the coordinate origin emits a photon with the wavelength λ forward. What wavelength measures a non-moving observer in the coordinate origin ?

Solution. We shall use directly the expression for the Doppler effect

$$\frac{\lambda'}{\lambda} = \frac{1}{\gamma(1 - v\cos\theta/c)}.$$

In our case the source is moving towards the receptor, so that $\theta = 180°$ and

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1-v/c}{1+v/c}}.$$

We need velocity which can be found as follows: $\gamma = E/mc^2$ and $v = c\sqrt{1-1/\gamma^2}$.

Problem 10.10. Potential energy (2D) is given by $U = A\cos\varphi/\rho^2$. A particle of the mass m is in the point $\boldsymbol{r} = (a, a)$ and its velocity is $\boldsymbol{v} \perp \boldsymbol{r}$. Find the normal acceleration.

Solution. By definition $\boldsymbol{a}_n \perp \boldsymbol{v}$. Since $\boldsymbol{r} \perp \boldsymbol{v}$ we have $\boldsymbol{a}_n \parallel \boldsymbol{r}$. Thus, the magnitude $a_n = |\boldsymbol{a} \cdot \hat{\boldsymbol{r}}|$.

a) The long way. $\hat{\boldsymbol{r}} = (a\hat{\boldsymbol{x}} + a\hat{\boldsymbol{y}})/\sqrt{2a^2} = (\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})/\sqrt{2}$. Express the potential energy in Cartesian coordinates $x = r\cos\varphi$, $y = \sin\varphi$: $U = Ax(x^2+y^2)^{-3/2}$. Then

$$\begin{aligned}
\boldsymbol{F} &= -\frac{\partial U}{\partial x}\hat{\boldsymbol{x}} - \frac{\partial U}{\partial y}\hat{\boldsymbol{y}} \\
&= -\left(\frac{A}{r^3} - \frac{3Ax^2}{r^5}\right)\hat{\boldsymbol{x}} + \left(\frac{3Axy}{r^5}\right)\hat{\boldsymbol{y}} \\
&= \frac{A(3\cos^2\varphi - 1)}{r^3}\hat{\boldsymbol{x}} + \frac{3A\sin\varphi\cos\varphi}{r^3}\hat{\boldsymbol{y}}.
\end{aligned}$$

Now $\boldsymbol{a} = \boldsymbol{F}/m$, $r = a\sqrt{2}$, and $\varphi = 45°$, so that $a_n = A/2ma^3$.

b) The short way. $\boldsymbol{F} = -(\partial U/\partial r)\hat{\boldsymbol{r}} - (1/r)(\partial U/\partial\varphi)\hat{\boldsymbol{\varphi}}$, so that $a_n = |(\partial U/\partial r)|/m = |2A\cos\varphi/r^3|/m = A/2ma^3$.

Problem 10.11. A satellite of the mass m is moving on an elliptical orbit round the Earth (mass $M \gg m$), so that $r_{max} = 2r_{min}$. The satellite energy $E < 0$ is known. Find the angular momentum.

Solution. When $r = r_{min}$ or $r = r_{max}$ the radial velocity vanishes $\dot{r} = 0$, so that in these points the energy and momentum conservation give

$$E = \frac{J^2}{2mr^2} - \frac{GMm}{r}.$$

Thus,

$$r_{1,2} = \frac{GMm \pm \sqrt{(GMm)^2 - 2|E|J^2/m}}{2|E|}$$

(here we used $E = -|E| < 0$) and

$$\frac{GMm + \sqrt{(GMm)^2 - 2|E|J^2/m}}{GMm - \sqrt{(GMm)^2 - 2|E|J^2/m}} = 2.$$

Solving this equation we get

$$J = \frac{2GMm\sqrt{m}}{3\sqrt{|E|}}.$$

Another way of solving: from the energy expression we have

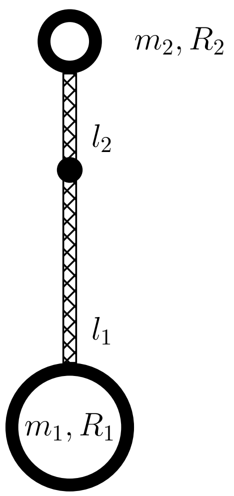
$$J^2/2m = r^2E + GMmr = (2r)^2E + GMm(2r),$$

so that $r = -GMm/3E$ and

$$J = \sqrt{2m(r^2E + GMmr)}$$

same as above.

Problem 10.12. Two disks of the masses m_1 and m_2 and radii R_1 and R_2 are connected to a massless rod which can rotate in the vertical plane. The disk centers are at the distances l_1 and l_2 from the rotation axis. Find the frequency of small oscillations.



Solution. If the system moved to the angle $\theta \ll 1$ and the angular velocity is $\dot{\theta}$, the energy is

$$E = \frac{I\dot{\theta}^2}{2} + mgL(1 - \cos\theta) = \frac{I\dot{\theta}^2}{2} + \frac{mgL\theta^2}{2},$$

where

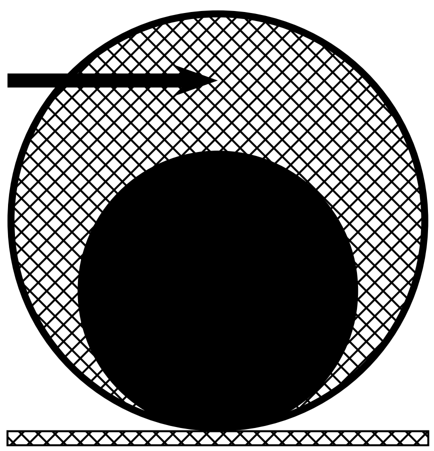
$$I = (\frac{m_1 R_1^2}{2} + m_1 l_1^2) + (\frac{m_2 R_2^2}{2} + m_2 l_2^2)$$

is the moment of inertia relative to the axis, and

$$L = \frac{m_1 l_1 - m_2 l_2}{m_1 + m_2}$$

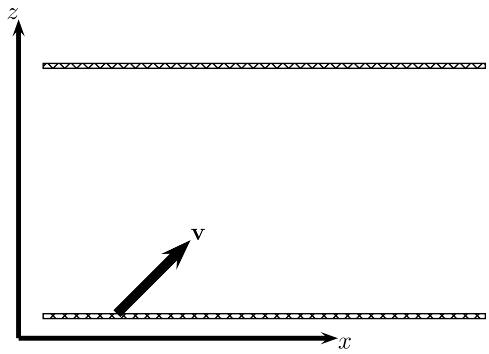
is the distance of the center of mass from the axis. The frequency is $\omega^2 = mgL/I$.

Problem 10.13. To a disk of the mass M and radius R a smaller disk of the mass mv and radius r is connected. At what height a horisontal force should be applied in order that the body start (in the very first moment) to roll without sliding. There is no friction.



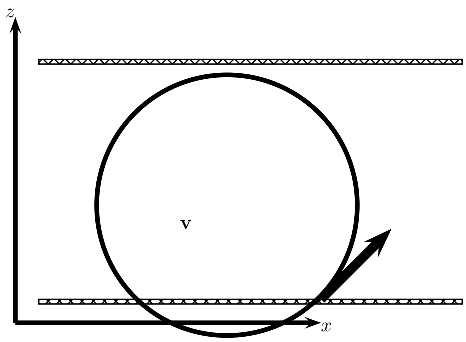
Solution. Let the force be F. The acceleration of the center of mass is $a = F/(M + m)$. The angular acceleration around the contact point is $\alpha = N/I = Fh/I$, where the moment of inertia is $I = (MR^2/2 + MR^2) + (mr^2/2 + mr^2) = (3/2)(MR^2 + mr^2)$. If there is no sliding, the linear and angular acceleration are related by $a = \alpha L$, where $L = (MR + mr)/(M + m)$ is the distance of the center of mass from the rotation axis (contact point). Thus, we have $F/(M + m) = (Fh/I)L$, so that $h = I/L(M + m)$.

Problem 10.14. There is a homogeneous magnetic field $\boldsymbol{B} = (0, B, 0)$ between two plates, parallel to the $x - y$ plane. A particle of the mass m and electric charge $q > 0$ enters the space between the planes through the lower plate, when its velocity is $\boldsymbol{v} = (v\cos 45°, 0, v\sin 45°)$. What is the minimal distance between the plates for which the particle will come back to the lower plate



(without touching the upper one) ? Gravity is negligible.

Solution. The particle moves on a circle of the radius r which is obtained from $F = qvB = mv^2/r$, that is, $r = mv/qB$, as shown in the figure.



From the figure it is clear that the distance between the two planes should be $L > r + r/\sqrt{2}$.

Problem 10.15. An observer on Earth sees one galaxy moving with the velocity $\boldsymbol{v}_1 = (0.5c, 0.5c, 0)$ and another with $\boldsymbol{v}_2 = (-0.5c, 0.5c, 0)$. Find their relative velocity $|\boldsymbol{v}_{rel}|$.

Solution. The easiest way is to rotate the coordinates so that the new x axis will be along $\boldsymbol{v}_1$, then the new y axis will be along $\boldsymbol{v}_2$. In the new coordinates $\boldsymbol{v}_1 = (0.5\sqrt{2}c, 0, 0)$, $\boldsymbol{v}_2 = (0, 0.5\sqrt{2}c, 0)$. Taking $\boldsymbol{v}_1 \equiv \boldsymbol{v}_0$ as the velocity of the moving frame (S'), and $\boldsymbol{v}_2$ as the velocity of a body we are looking at, we have

$$v'_x = \frac{v_x - v_0}{1 - v_x v_0/c^2} = -v_0 = -0.5\sqrt{2}c,$$

$$v'_y = \frac{v_y}{\gamma(1 - v_x v_0/c^2)} = \frac{v_y}{\gamma} = \frac{0.5\sqrt{2}c}{\gamma}.$$

where we used $v_x = (\boldsymbol{v}_2)_x = 0$, and

$$\gamma = (1 - v_1^2/c^2)^{-1/2} = \sqrt{2}.$$

Thus, we have $v'_x = -0.5\sqrt{2}c$, $v'_y = 0.5c$, and $v' = \sqrt{{v'_x}^2 + {v'_y}^2} = 0.5\sqrt{3}c$.

Problem 10.16. Potential energy is given as $U = -A/r^4$, $A > 0$. When the particle (mass m) is in the point $(a, 0, 0)$ its velocity is $\boldsymbol{v} = (0, v, 0)$ and its energy is negative. What is the maximum distance between the particle and the coordinate origin ?

Solution. Central force means that $\boldsymbol{J} = m\boldsymbol{r}\times\boldsymbol{v} = \text{const}$ and $E = mv_r^2/2 + J^2/2mr^2 + U = \text{const}$. From the initial conditions we have $\boldsymbol{J} = J\hat{\boldsymbol{z}}$, $J = mav$, $E = mv^2/2 - A/a^4$. Since $E < 0$ we have $v^2 < 2A/ma^4$. In the closest and farthest points $v_r = 0$ so that we have

$$\frac{J^2}{2mr^2} - \frac{A}{r^4} = E$$

where J, m, $E < 0$, and A are known. The equation is easily solved to give

$$r^2 = -\frac{J^2}{4m|E|} \pm \sqrt{(\frac{J^2}{4m|E|})^2 + \frac{A}{|E|}}$$

The solution with $-$ is not physical ($r^2 < 0$) which means that there is no minimum radius: the particle reaches the maximum radius, bounces back and falls into $r = 0$.

Problem 10.17. A particle moves according to $x = r_0\cos(\omega t)$, $y = r_0\sin(\omega t)$, $z = kt^2/2$. Find the normal acceleration.

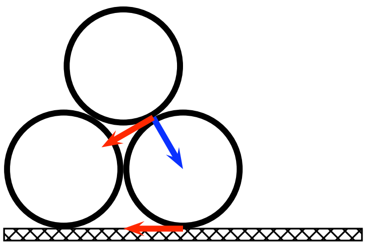
Solution.

$$\begin{aligned}
\boldsymbol{v} &= \omega r_0(-\sin(\omega t)\hat{\boldsymbol{x}} + \cos(\omega t)\hat{\boldsymbol{y}}) + kt\hat{\boldsymbol{z}} \\
\boldsymbol{a} &= -\omega^2 r_0(\cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}}) + k\hat{\boldsymbol{z}} \\
a_n &= |\boldsymbol{a}\times\hat{\boldsymbol{v}}| \\
&= \omega r_0\frac{\sqrt{\omega^2k^2t^2 + k^2 + \omega^4r_0^2}}{\sqrt{\omega^2r_0^2 + k^2t^2}}
\end{aligned}$$

Problem 10.18. Potential energy is $U(x) = A(x-a)^2(x-b)^2$, $A > 0$. Find the frequency of small oscillations.

Solution. Equilibrium: $dU/dx = 0 \to x = a$ or $x - b$. It is easy to find that $U = A(b-a)^2X^2$ where $X = x - a$ or $X = x - b$. Thus, $\omega^2 = 2A(b-a)^2/m$.

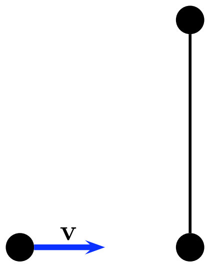
Problem 10.19. Three identical cylinders are in equilibrium (see figure). Find the minimum friction coefficient between the cylinders.



Solution. Let the normal force acting on the lower cylinder from the upper is N, and the friction force between the upper and lower cylinders is f_s, $f_s/N \le \mu$. From the torque balance on the lower cylinder the friction force between the lower cylinder and the table is f_s. The force balance on the lower cylinder in the horisontal direction is

$$f_s + f_s \sin 30^\circ = N \sin 60^\circ \to \mu \ge \frac{f_s}{N} = \sqrt{3}/3$$

Problem 10.20. A body is built of two small identical balls connected with a massless rod of the length l. Initially the body is at rest. Another identical ball, moving with the velocity v, collides elastically with one of these two (see figure). Find the angular velocity of the body rotation after the collision.



Solution. Elastic collision means that the momentum, angular momentum, and energy are conserved. Let the velocity of the third ball after the collision be v' (it will be in the same direction as the initial velocity or in the opposite direction - in the last case $v' < 0$). Let the velocity of the center of the rod be V and the angular velocity of the rotation around this center (counterclockwise) be ω. Momentum conservation gives

$$mv = mv' + 2mV$$

Angular momentum conservation (relative to the collision point):

$$0 = 2mV(l/2) - 2m(l/2)^2\omega$$

Energy conservation gives

$$\frac{mv^2}{2} = \frac{mv'^2}{2} + \frac{2mV^2}{2} + \frac{2m(l/2)^2\omega^2}{2}$$

Solving these equations we obtain $\omega = v/l$.

Problem 10.21. In a box of the size $a \times b \times c$ (in its rest frame) N particles are homogeneously distributed. Find the particle density as viewed by an observer moving with the velocity v along a.

Solution. $a' = a/\gamma$, $V' = V/\gamma \to n' = n\gamma$