Lecture Notes on Special Relativity

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The Relativistic Principle

1 Poincaré

One of the most fruitful ideas that is currently used in physics originated with Albert Einstein is the concept that the laws of physics should take the same form in all inertial frames and that the speed of light is the same for all inertial observers. Thus if the light wave originates at the origin of a coordinate system X at time, $t = 0$ then the light front at a later time is a spherical wave whose radius is given by ct where c is the velocity of light for any observer. That is,

$$(ct)^2 = \boldsymbol{x}^2 \tag{1}$$

However, an observer in a frame, X' (say moving at velocity v along the x-axis), also sees the light wave originating at origin of his/her coordinates at $t' = 0$, and will see a spherical wave of radius ct' where

$$(ct')^2 = \boldsymbol{x'}^2 \tag{2}$$

emanating from the center of his/her frame, X' moving relative to the X frame at a constant velocity v. That these observers can see a spherical wave emanating from the center of each of their coordinate systems can be shown to be consistent if the transformations that connect the coordinates of each observer are the so called Lorentz transformations which we now define. In general we can assume that [1]

$$(ct)^2 - \boldsymbol{x}^2 = \eta((ct')^2 - \boldsymbol{x'}^2) \tag{3}$$

where we have included a scale factor η as the condition that spherical surfaces of light waves appear the same for each observer do not determine this factor. We will now see that η must equal 1. First of all, η must only depend on the absolute value of $\mathbf{v}$ and not on its direction. Let us consider two independent transformations, one in the positive x direction and one in the negative one. For the first transformation in the $+x$ directionwe have

$$(ct')^2 - \mathbf{x}'^2 = \eta(|\mathbf{v}|)((ct)^2 - \mathbf{x}^2) \tag{4}$$

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$$(ct'')^2 - \mathbf{x}''^2 = \eta(|\mathbf{v}|)((ct')^2 - \mathbf{x}'^2) \tag{5}$$

[1] We use the following modification to account for scale change- see E. Abers' notes.

Since the transformation of Eq. 5 cancels that first transformation of Eq. 4 we have

$$(ct)^2 - \mathbf{x}^2 = \eta(|\mathbf{v}|)((ct')^2 - \mathbf{x}'^2) = \eta(|\mathbf{v}|)^2((ct)^2 - \mathbf{x}^2) \tag{6}$$

which implies that $\eta^2 = 1$ or $\eta = \pm 1$. An infinitesimal boost should lead to a infinitesimal change in our expression. This means that only the positive sign makes any sense. We henceforth take $\eta = 1$. Thus we now write for Eq. 4

$$(ct')^2 - \mathbf{x}'^2 = (ct)^2 - \mathbf{x}^2 \tag{7}$$

Therefore we now take

$$x' = \gamma(x - vt) \qquad \text{and} \qquad t' = At + Bx \tag{8}$$

then if we substitute Eq. 8 into Eq. 2 and use Eq. 1 we find that [**?**]

$$x' = \gamma(x - vt) \qquad \text{and} \qquad t' = \gamma(t - \beta\frac{x}{c}) \tag{9}$$

where $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}$ and $\beta = \frac{v}{c}$. Here, in Eq. 8, we have assumed linearity and that the point $x' = 0$ is described in the original frame by the equation $x = vt$. More generally, we could consider the two observers moving at an arbitrary direction from each other. We could then relate the result obtained above by rotating the X' coordinate system under the orientation of this velocity to obtain the transformations for the more general case.

Alternatively [2]to be more explicit, we can set

$$c^2t'^2 - x'^2 = c^2t^2 - x^2$$

where we consider $\mathbf{v}$ along the x-direction so that $y' = y \;\; z' = z$ we can write

$$(ct'+x')(ct'-x') = (ct+x)(ct-x) \Longrightarrow ct'+x' = \lambda^{-1}(ct+x), \;\; ct'-x' = \lambda(ct-x)$$

where λ is to be determined. It is convenient to write λ in terms of a parameter θ as $\lambda = e^{\theta}$. From these last two equations we have

$$ct' = \cosh(\theta)ct - \sinh(\theta)x, \quad x' = -\sinh(\theta)ct + \cosh(\theta)x$$

. The point $x' = 0$ moves with a velocity v with respect to the S-frame. Thus

$$\tanh(\theta) = \frac{x}{ct} = \frac{v}{c}$$

and

$$\cosh(\theta) = \frac{1}{\sqrt{1 - \tanh(\theta)^2}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma$$

and

$$\sinh(\theta) = \tanh(\theta)\cosh(\theta) = \beta\gamma$$

[2]L.Parker, G.M. Schmieg, *Am .J. Phys.* **38**, 218 (1970)

A Poincaré transformation includes, besides the Lorentz transformation described above, a translation as well. Before proceeding further, it is useful to use tensor notation. Thus we designate time and space with one symbol that can be referred to as a four vector. In particular, what we mean is given by the following:

$$x^\mu = (ct, x, y, z) \qquad \text{or} \qquad x^0 = ct, \quad x^1 = x, \quad x^2 = y \quad x^3 = z$$

In this notation we can write Eq. 9 in the form $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ where

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

It will prove useful to define the metric tensor $g_{\mu\nu}$ having the form given by the matrix

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{11}$$

We define $g^{\mu\nu}$ by $g^{\mu\nu} \equiv g^{-1}_{\mu\nu}$. This can be taken as a general definition of $g^{\mu\nu}$ even for space-time dependent $g_{\mu\nu}$. In this particular case $g_{\mu\nu} = g^{\mu\nu}$ which in general is not true.

We use $g_{\mu\nu}$ to define x_μ by $x_\mu = g_{\mu\nu} x^\nu$ from where we see that

$$x_\nu = (ct, -x, -y, -z) \tag{12}$$

With this notation, we can write

$$x^\mu x_\mu = (ct)^2 - \boldsymbol{x}^2 \tag{13}$$

where we have summed over repeated indices. Furthermore, it is easily seen that $x^\mu x_\mu$ is invariant under the transformation Eq. 9, even when $x^\mu x_\mu \neq 0$. This is a consequence of the fact that the matrix $\Lambda^\mu{}_\nu$ (i.e., Eq. 10) satisfies the equations

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda = g_{\rho\lambda}; \qquad \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \, g^{\alpha\beta} = g^{\mu\nu} \tag{14}$$

More generally, a Poincaré transformation consists of a boost, a rotation and a translation. The boost relates two inertial frames moving relative to each other with a velocity $\boldsymbol{v}$. A special case of this is given by Eq. 9 for a boost along the x-axis. For a generic Lorentz transformation, Eq. 14 will still be satisfied. We will refer to this condition as the generic Lorentz transformation.

A general Lorentz transformation (i.e., Poincaré transformation) can be written as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{15}$$

where Λ is a linear transformation depending on the velocity $(\boldsymbol{v})\, between two inertial frames.$

——————————————————————————— Previously we have found the Lorentz transformation, $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ where $\Lambda^{\mu}{}_{\nu}$ is given by Eq. 10. This was for the case that the relative motion between the frames was along the $x-$axis. Find the expression for the Lorentz transformation for the case in which the relative velocity $\mathbf{v}$ is arbitrary vector (not confined to be along the $x-$axis) with $|\mathbf{v}| \leq c$.

(Hint: we have for the origin of the coordinates in original frame (i.e., in the unprimed frame) that $dx_i = 0$ therefore we have

$$\begin{aligned} dx'^{i} &= \Lambda^{i}{}_{0} dt \\ dt' &= \Lambda^{0}{}_{0} dt \end{aligned} \tag{16}$$

Dividing one of these equations by the other, we have $v^i \Lambda^0{}_0 = \Lambda^i{}_0$. Since $\frac{dx'^i}{dt'} = v^i$ is the velocity of the frame S as observed from S'. From first in equation in Eq. 14, show that for $\rho, \lambda = 0$ that $\Lambda^0{}_0 = \frac{1}{\sqrt{1-\mathbf{v}^2}}$, where we have taken $c = 1$ therefore $\Lambda^i{}_0 = v^i \gamma$. Finally, solve for the rest of the matrix elements Λ^i_j and Λ^0_j using the the ansatz:

$$\begin{aligned} \Lambda^{i}{}_{j} &= A\delta^i_j + B v^i v^j \\ \Lambda^{o}{}_{j} &= C v^j \end{aligned}$$

find A,B, C.

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Objects that transform under a change of reference frame as homogeneous Lorentz transformation such a differentials of the coordinates are referred to as four-vector. That is, in the above equation $dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu}$ Notice in this expression, the indices on the x's are superscript. These x's with superscripts are called contra-variant objects. One can use use the $g^{\mu\nu}$'s to define covariant vectors from contra-variant by $A_{\mu} = g_{\mu\nu} A^{\nu}$.

To see how the gradient, $\frac{\partial}{\partial x^{\mu}}$ transforms we can use Eq. 15 where we find

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial}{\partial x'^{\lambda}} = \Lambda^{\lambda}{}_{\mu} \frac{\partial}{\partial x'^{\lambda}} \tag{17}$$

From this we solve for $\frac{\partial}{\partial x'^{\lambda}}$. Using the Lorentz condition, i.e., the second equation of Eq. 14, we can write $\Lambda^{\mu}{}_{\alpha}(g^{\alpha\beta} \Lambda^{\nu}{}_{\beta} g_{\nu\kappa}) = g^{\mu\nu} g_{\nu\kappa} = \delta^{\mu}{}_{\kappa}$ where we can define $\Lambda_{\kappa}{}^{\alpha} = g^{\alpha\beta} \Lambda^{\nu}{}_{\beta} g_{\nu\kappa}$. We note

$$\Lambda_{\kappa}{}^{\alpha} \Lambda^{\mu}{}_{\alpha} = \delta^{\mu}{}_{\alpha} \tag{18}$$

We can now use this equation to write $\Lambda_{\theta}{}^{\mu} \frac{\partial}{\partial x^{\mu}} = \Lambda_{\theta}{}^{\mu} \Lambda^{\lambda}{}_{\mu} \frac{\partial}{\partial x'^{\lambda}}$. Thus

$$\frac{\partial}{\partial x'^{\theta}} = \Lambda_{\theta}{}^{\mu} \frac{\partial}{\partial x^{\mu}} \tag{19}$$

We will now show that any covariant vector, A_{μ} defined from a contra-variant vector by $A_{\mu} = g_{\mu\nu} A^{\nu}$ transforms as a gradient, i.e., $A'_{\mu} = g_{\mu\nu} A'^{\nu} =$

$g_{\mu\nu}\Lambda^{\nu}{}_{\beta}A^{\beta} = g_{\mu\nu}\Lambda^{\nu}{}_{\beta}g^{\beta\kappa}A_{\kappa} = \Lambda_{\mu}{}^{\kappa}A_{\kappa}$. We shall always use the convention that upper indicies transform contravariantly while lower ones, covariantly.

It is useful to note, that the D'Alembertian is a scalar with respect to a Lorentz transformation.

Furthermore, since $\equiv \partial^2 = g^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}$

$$= g^{\mu\nu}\Lambda^{\lambda}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\frac{\partial}{\partial x'^{\lambda}}\frac{\partial}{\partial x'^{\sigma}} \tag{20}$$

Using the defining condition of the Lorentz transform, Eq. 14, the above equation, i.e., Eq. 20 becomes

$$= g^{\lambda\sigma}\frac{\partial}{\partial x'^{\lambda}}\frac{\partial}{\partial x'^{\sigma}} \tag{21}$$

which shows that the D'Ambertian is invariant under a Lorentz transformation.

1.1 Tensors with an Arbitrary of Up and Down Indicies

We define a tensor with r upper indicies and s lower indicies $T^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s}$ by its transformation properties as follows:

$$T'^{\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_s} = \Lambda^{\alpha_1}{}_{\gamma_1}\cdots\Lambda^{\alpha_r}{}_{\gamma_r}\Lambda_{\beta_1}{}^{\delta_1}\cdots\Lambda_{\beta_s}{}^{\delta_s}T^{\gamma_1\ldots\gamma_r}{}_{\delta_1\ldots\delta_s} \tag{22}$$

Operations with Tensors

i) <u>Linear combination</u> of tensors of the same type, i.e.,
$T^{\alpha}{}_{\beta} \equiv aR^{\alpha}{}_{\beta} + bS^{\alpha}{}_{\beta}$
where a and b are scalars. We can see immediately that T^{α}_{β} is a tensor since
$T'^{\alpha}_{\beta} \equiv aR'^{\alpha}_{\beta} + bS'^{\alpha}_{\beta} = a\Lambda^{\alpha}_{\gamma}\Lambda_{\beta}{}^{\delta}R^{\gamma}_{\delta} + b\Lambda^{\alpha}_{\gamma}\Lambda_{\beta}{}^{\delta}S^{\gamma}_{\delta} = \Lambda^{\alpha}_{\gamma}\Lambda_{\beta}{}^{\delta}T^{\gamma}_{\delta}$

ii) Direct product of tensors, i.e.,
$T^{\alpha}{}_{\beta}{}^{\gamma} = A^{\alpha}_{\beta}B^{\gamma}$ where A^{α}_{β} and B^{γ} are tensors.Then $T^{\alpha}{}_{\beta}{}^{\gamma}$ is a tensor since
$T'^{\alpha}{}_{\beta}{}^{\gamma} = A'^{\alpha}_{\beta}B'^{\gamma} = \Lambda^{\alpha}{}_{\delta}\Lambda_{\beta}{}^{\epsilon}\Lambda^{\gamma}{}_{\tau}T^{\delta}{}_{\epsilon}{}^{\tau}$

iii) Contraction of a tensor
We can construct the tensor $T^{\alpha\gamma} \equiv T^{\alpha}{}_{\beta}{}^{\gamma\beta}$. However we see that
$T'^{\alpha\gamma} = T'^{\alpha}_{\beta}{}^{\gamma\ \beta} = \Lambda^{\alpha}{}_{\delta}\Lambda_{\beta}{}^{\epsilon}\Lambda^{\gamma}{}_{\tau}\Lambda^{\beta}{}_{\kappa}T^{\delta}{}_{\epsilon}{}^{\tau\kappa}$ but since $\Lambda_{\beta}{}^{\epsilon}\Lambda^{\beta}{}_{\kappa} = \delta^{\epsilon}_{\kappa}$ we set
$T'^{\alpha\gamma} = \Lambda^{\alpha}{}_{\delta}\Lambda^{\gamma}_{\tau}\delta^{\epsilon}_{\kappa}T^{\delta}{}_{\epsilon}{}^{\tau\kappa}$
$T'^{\alpha\gamma} = \Lambda^{\alpha}{}_{\delta}\Lambda^{\gamma}{}_{\tau}T^{\delta\tau}$

iv) Differentiation
The derivative, $\frac{\partial}{\partial x^{\alpha}}$ of any tensor is a tensor with an additional index' i.e., if $T^{\beta\gamma}$ is a tensor, then $T_{\alpha}{}^{\beta\gamma} = \frac{\partial}{\partial x^{\alpha}}T^{\beta\gamma}$ is also a tensor.
Proof:
$T'_{\alpha}{}^{\beta\gamma} = \frac{\partial}{\partial x'^{\alpha}}T'^{\beta\gamma} = \Lambda_{\alpha}{}^{\delta}\frac{\partial}{\partial x^{\delta}}\Lambda^{\beta}{}_{\epsilon}\Lambda^{\gamma}{}_{\tau}T^{\epsilon\tau} = \Lambda_{\alpha}{}^{\delta}\Lambda^{\beta}{}_{\epsilon}\Lambda^{\gamma}{}_{\tau}\frac{\partial}{\partial x^{\delta}}T^{\epsilon\tau} = \Lambda_{\alpha}{}^{\delta}\Lambda^{\beta}_{\epsilon}\Lambda^{\gamma}_{\tau}T_{\delta}{}^{\epsilon\tau}$

2 Relativistic Dynamics

2.1 Principle of Least Action

The action, S of this particle can be represented by $S = -\alpha \int ds$ where $ds = \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g_{\mu\nu}}d\lambda$ where λ, a function of time, is a monotonic increasing parameter and ds is the proper length between two arbitrary events along the particles trajectory and α is a constant of proportionality. Choosing λ to be the time, t, the following expression is obtained for the action:

$$S = -\alpha c \int \sqrt{1 - \frac{v^2}{c^2}}dt \tag{23}$$

with $v = \sqrt{\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt}}$. The proportionality factor, α is found by the square root factor in Eq. 23 and comparing with its non-relativistic counter-part where $L = \frac{mv^2}{2}$. That is

$$-\alpha c\sqrt{1 - \frac{v^2}{c^2}} \approx -\alpha c + \alpha c\frac{v^2}{2c^2}$$

Therefore we see that $\alpha = mc$. We therefore have

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} \tag{24}$$

2.2 Energy and Momentum

As previously defined, momentum $\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$ which leads directly (using 24) to

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The Hamiltonian, $H = \mathbf{p} \cdot \dot{\mathbf{x}} - L$ becomes

$$H = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{25}$$

which for the free point particle is identical to the conserved energy, E since the Lagrangian is time-independent.

Energy and momentum of a point particle can be combined into a four-vector p^μ with $p^\mu \equiv (E/c, \mathbf{p})$ where $p^\mu = m\frac{dx^\mu}{d\tau}$, $\quad d\tau^2 = dt^2 - \frac{d\mathbf{x}^2}{c^2}$
$p^0 = mc\frac{dt}{d\tau} = mc\gamma$, $\quad \mathbf{p} = m\mathbf{v}\gamma$, $\quad p^\mu p_\mu = (\frac{E}{c})^2 - \mathbf{p}^2 = m^2c^2$ where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

2.3 External Electromagnetic Field

One of the main applications of relativistic quantum mechanics is to systems comprised of charged particles either interacting with each other or in the presence of an external electromagnetic field. In this section we discuss the way this interaction is incorporated in our Lagrangian formalism. Let us consider the situation in which electromagnetic potential A_μ is present. To incorporate its influence on charged point particles, we wish to modify the action in such a way that gauge invariance is preserved. This can be done done by adding the term S_{EM} where

$$S_{EM} = -e \int A_\mu dx^\mu = e \int A_\mu \frac{dx^\mu}{d\lambda} d\lambda \tag{26}$$

------------------ Show that $e \int A_\mu dx^\mu$ is invariant under the gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$$

where Λ is arbitrary function of x^μ ------------------ - ------------------ — Obtain the equations of motion for point particle of charge e when this interaction is added to the action given by Eq. 23, and show the equations of motion give the Lorentz force

$$m \frac{d^2 x^\mu}{ds^2} = e \frac{dx_\nu}{ds} F^{\mu\nu}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the Maxwell tensor. In terms of the Maxwell tensor $F^{\mu\nu}$, $F^{i0} = E^i$ and $F^{ij} = -\epsilon^{ijk} B^k$