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SOME CELESTIAL MECHANICS

The effect of the Earth's elliptical orbit.

The Earth's elliptical orbit, relative to the sun, causes the effective solar constant to vary slightly throughout the year. This variation is well approximated by the empirical formula:

$$S_{o,eff} = [1 + 0.033 \cos(360^{o} n / 365.25)] S_o \qquad \textbf{(2.1)}$$

where S_o = 1360 W/sq.m and n is the *Julian Day* number. [n=1 for January 1st. n=32 for February 1st. n=60 for March 1st (but n=61 for March 1st in leap years), etc]. One notices that eq. (**2.1**) takes its maximum value at the end of the year - about 10 days after the northern hemisphere's *Winter Solstice* (approx. December 21). The consequence of this is that southern hemisphere locations receive slightly more radiation in their summer months than do northern hemisphere sites, at similar latitudes, in theirs.

This elliptical motion also causes the Earth's tangential velocity to speed up as the planet approaches the sun and to slow down as it moves further out. This, together with the annual precession of the earth's axis of rotation, as viewed from the sun, gives rise to the sun not appearing at the highest point in the sky at noon on successive days. Thus the *solar time* given by a simple sundial will oscillate in a complicated manner relative to regular clock time. The discrepancy between solar time and standard time is referred to as the *Equation of Time.* It varies throughout the year between about -15 min in mid February to about +15 min at the end of October (**Fig. 1**).

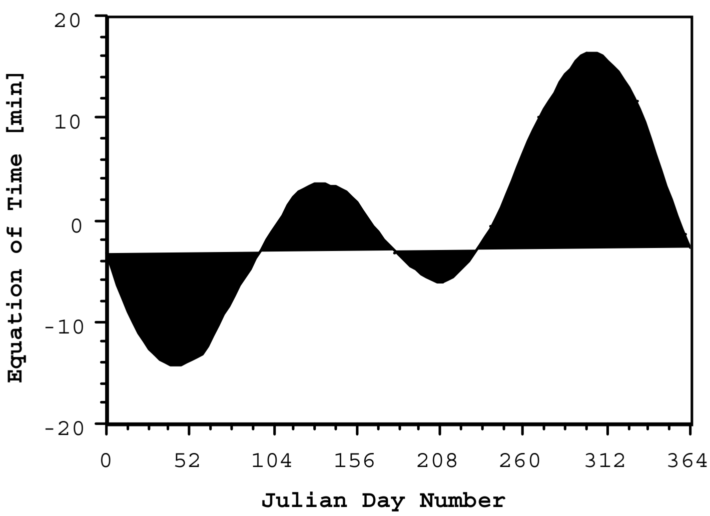

Figure 1: The Equation of Time

Accurate values of the Equation of Time are tabulated annually in astronomical almanacs but approximate values, of sufficient accuracy for many solar energy purposes, are given by the simple expression :

$$E = A \sin [C(n - n_1)] + B \sin [2C(n - n_2)] \qquad \textbf{(2.2)}$$

where, $A = -7.3584$ min, $B = 9.927$ min,
$C = 2\pi/365.25$, $n_1 = 4.41$, $n_2 = 81.97$, **(2.3)**

The two terms in eq. (**2.2**) represent the respective contributions from the ellipticity of the earth's orbit and the tilt of its axis. This approximation may differ from the true equation of time by up to half-a-minute at certain times of the year. A more accurate expression is given in the Appendix.

Apart from the corrections eqs. (**2.1**) and (**2.2**) for the ellipticity of the earth's orbit, acceptable accuracy - for most solar energy conversion calculational purposes - is achieved if we regard the Earth's motion as constituting a circle centered at the sun. By employing the Equation of Time together with a knowledge of the longitude of the site, local clock time can be converted to solar time - a convenient parameter for use in the quantitative design and evaluation of solar energy conversion systems. The conversion formula is:

$$\text{Solar Time} = \text{Standard Time} + 4(L_{st} - L_{loc}) + E \qquad \textbf{(2.4)}$$

where *Standard Time* usually equals local clock time in winter but may need to be corrected in summer if *daylight saving time* is in use. L_{st} is the standard longitude for the *time zone* in question (reckoned as positive west of the Greenwich meridian), L_{loc} is the true longitude of the site for which the calculation is to be made and E is the Equation of Time correction. Note that eqs. (**2.2**) and (**2.4**) are in minutes.

Example 1: *Calculate the time at which the sun reaches its highest point "solar noon" in the sky above Beersheba on November 9th 2003.*

Solution: (a) Israel is in the time zone represented by the standard longitude $L_{st} = -30^o$. The longitude of Beersheba is $L_{loc} = -34.8^o$. Thus, the longitude correction in eq. (**2.4**) is:
4[(-30) - (-34.8)] = 19.2 minutes. (b) November 9th 2003 corresponds to julian day n = 313. Therefore, from eqs. (**2.3**) and (**2.2**), E = 16.0 minutes. Thus, solar noon is at approximately 11:25 am (Standard Time).

SOLAR GEOMETRY

The following angles are commonly used to specify the apparent position of the sun in the sky:

hour angle ω

$\omega = 360^o (t - 12) / \tau_{day}$ **(2.5)**

where τ_{day} = 24 hr and t is the solar time (in hours) for which the hour angle is to be computed. Note that the hour angle is zero at solar noon, negative in the morning and positive in the afternoon.

Example 2: *Calculate the hour angle that corresponds to 12 noon (standard time) in Beersheba on November 9th 2003.*

Solution: (a) From eq. (**2.4**) and similar considerations to those in **Example 1**, the corresponding solar time is: t = 12 hrs + 19.2 min + 16.0 min = 12.5867 hrs. (b) From eq. (**2.5**), $\omega = +8.8^0$

declination angle δ

This is the angle between the earth's equatorial plane and the earth-sun vector. It is given by:

$\sin\delta = -\sin 23.45^o \cos[360^o(n + 10)/365.25]$ **(2.6)**

the derivation of which will be given below. From eq.(**2.6**) δ is seen to take its extreme values of $\pm 23.45^o$ at the so called *equinoxes,* (approximately March 21 and September 21).

Example 3: *Calculate the solar declination on November 9th 2003*.

Solution: For this date n = 313. Hence, eq. (**2.6**) gives: $\sin\delta = -0.2974$. Thus, $\delta = -17.3^0$.

To derive eq.(**2.6**) it is convenient to choose a Cartesian coordinate system centered on the sun, with the z-axis normal to the plane of the earth's orbit and the x-axis pointed at the position of the earth on December 21st, the unit vector pointing from the earth to the sun has components:

(2.7)

$$\hat{\mathbf{n}}_s = -(\cos\Omega, \sin\Omega, 0)$$

where the angle defining the earth's position in its orbit is:

$$\Omega = \frac{360^{\circ}(n + 10)}{365.25}$$

(2.8)

The unit vector normal to the earth's equatorial plane has components:

$$\hat{\mathbf{n}}_p = (\sin 23.45^{\circ}, 0, \cos 23.45^{\circ})$$

(2.9)

Since the solar declination δ is defined as the angle between the earth-sun vector and the earth's equatorial plane, the scalar product of the two unit vectors (**2.7**) and (**2.9**) is simply $\cos(90^0 - \delta)$ or sin δ. Eq.(**2.6**) is the result.

Three more angles are generally needed to describe the manner in which a beam of solar radiation is incident on a solar collector: λ the geographic *latitude* of the site (positive in the northern hemisphere, negative in the south), β the *tilt angle* made by the normal to the collector aperture plane and the zenith, and ϕ the *azimuth angle* between the north-south direction and a plane at right angles to the collector's axis of tilt. By careful choice of Cartesian axes - appropriate to each case in question - the trigonometric formulae of interest in the design of energy conversion systems can usually be obtained by simple vector algebra [**2**].

Incidence angles on various surfaces

If $\hat{\mathbf{n}}_c$ is a unit vector normal to some surface of interest - e.g. the plane of a window or of a solar collector, and $\hat{\mathbf{n}}_s$ is a unit vector directed from the earth to the sun, and referred to the same coordinate system - then θ, the angle of incidence of a solar beam striking the surface, is given by:

$$\hat{\mathbf{n}}_c \bullet \hat{\mathbf{n}}_s = \cos\theta$$

(2.10)

By selecting the rectangular Cartesian coordinate system appropriate to each case of interest, eq.(**2.10**) can be used to derive the angle of incidence for solar collectors of any desired orientation (either fixed or sun-tracking). Some important results are listed in **Table 1** [**2**]:

TRACKING MODE AND ORIENTATION	$\cos\theta$
Non-tracking, azimuth ϕ, tilt β	$\cos\delta\cos\lambda\cos\beta\cos\omega$ $+\ \sin\delta\sin\lambda\cos\beta$ $+\ \cos\delta\sin\phi\sin\beta\sin\omega$ $+\ \cos\delta\cos\phi\sin\lambda\sin\beta\cos\omega$ $-\ \sin\delta\cos\lambda\sin\beta\cos\phi$
E -> W tracking about a NS horizontal axis	$\cos\delta\,[\,\sin^2\omega + (\cos\lambda\cos\omega + \tan\delta\sin\lambda)]^{1/2}$
Tracking about an EW horizontal axis	$\cos\delta\,(\,\cos^2\omega + \tan^2\delta)^{1/2}$
Tracking about a fixed polar axis	$\cos\delta$
Full dual-axis tracking	1

Table 1: Solar incidence angles for some commonly encountered collector orientations and tracking modes

The term "polar axis" in **Table 1** refers to a collector that tracks the sun while rotating about a fixed axis parallel to that of the earth's rotation axis.

Example 4: *Compute the angle of incidence on a fixed surface of arbitrary orientation (i.e. the first entry in **Table 1)**.*

Solution: For this purpose it is convenient to use the vector method of reference [**2**]. We start with the right-handed rectangular Cartesian reference frame shown in **Fig. 2**, fixed to the earth at the location of the collector, in which the x-axis points towards the noon-time sun at equinox, the y-axis points eastwards and the z-axis is parallel to the earth's axis of rotation.

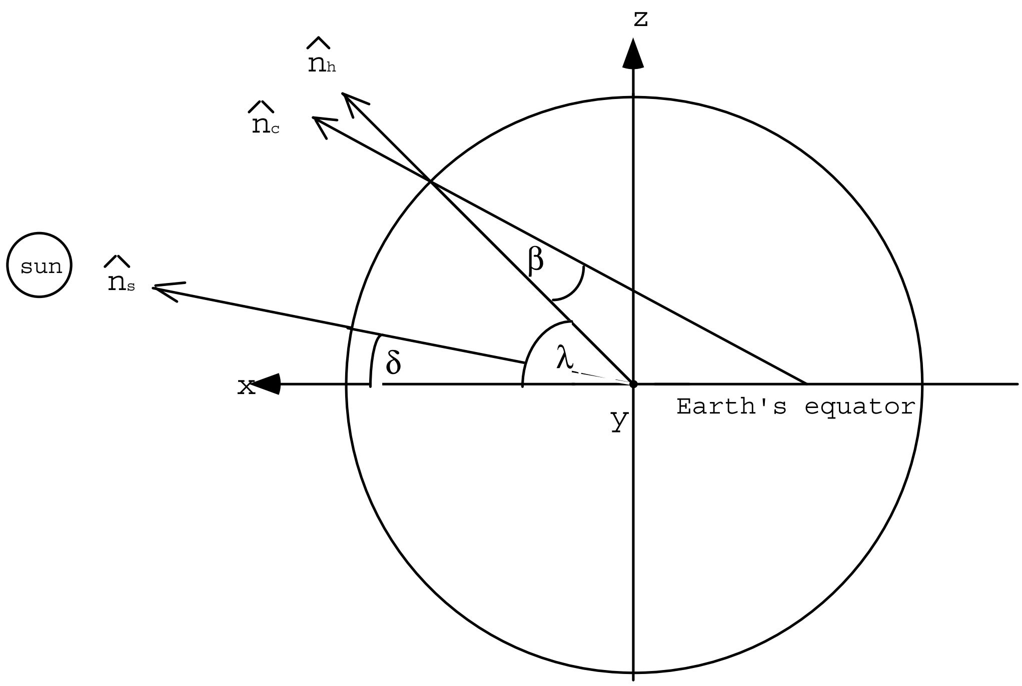

Figure 2: Cartesian coordinate system convenient for deriving the results in Table 1

In such a frame the components of a unit vector pointing from the earth to the sun are:

$$n_{s,x} = \cos\delta \ \cos\omega\ , \ \ n_{s,y} = -\cos\delta \ \sin\omega\ , \ \ n_{s,z} = \sin\delta \qquad \textbf{(2.11)}$$

We first give the coordinate system a *negative* rotation (in the northern hemisphere) through an angle λ about the y-axis. [In the southern hemisphere the same equations effect a positive rotation of axes through an angle $-\lambda$ about the y-axis]. The new $x^{(1)}$-axis now points towards the local *zenith* (i.e. vertically upwards), the new $y^{(1)}$ axis is coincident with the old y axis and the new $z^{(1)}$-axis points due north. The effect of this rotation of axes is to give the earth-sun unit vector the new components:

$$\begin{aligned} n_{s,x}^{(1)} &= n_{s,z} \sin\lambda + n_{s,x} \cos\lambda \\ n_{s,y}^{(1)} &= n_{s,y} \\ n_{s,z}^{(1)} &= n_{s,z} \cos\lambda - n_{s,x} \sin\lambda \end{aligned} \qquad \textbf{(2.12)}$$

Second, we rotate the resulting coordinate system through an angle $+\phi$ about the $x^{(1)}$-axis. [The phase convention for ϕ is: $\phi = 0$ represents a collector facing due south, $\phi = 90^0$ represents an eastward-facing collector, $\phi = -90^0$ represents a westward-facing collector, etc.]. The earth-sun unit vector now has the components:

$$\begin{aligned} n_{s,x}^{(2)} &= n_{s,x}^{(1)} \\ n_{s,y}^{(2)} &= n_{s,y}^{(1)} \cos\phi + n_{s,z}^{(1)} \sin\phi \\ n_{s,z}^{(2)} &= -n_{s,y}^{(1)} \sin\phi + n_{s,z}^{(1)} \cos\phi \end{aligned} \qquad \textbf{(2.13)}$$

At this stage the new $y^{(2)}$ and $z^{(2)}$ axes are both horizontal, the new $x^{(2)}$ axis coinciding with the previous $x^{(1)}$ axis. Our third rotation of coordinates is through an angle β about the $y^{(2)}$-axis which will leave this axis horizontal. The earth-sun unit vector now receives the components:

$$\begin{aligned} n_{s,x}^{(3)} &= -n_{s,z}^{(2)} \sin\beta + n_{s,x}^{(2)} \cos\beta \\ n_{s,y}^{(3)} &= n_{s,y}^{(2)} \\ n_{s,z}^{(3)} &= n_{s,z}^{(2)} \cos\beta + n_{s,x}^{(2)} \sin\beta \end{aligned} \qquad \textbf{(2.14)}$$

In this new coordinate frame the $x^{(3)}$-axis is normal to the collector plane and the $y^{(3)}$-axis remains horizontal. Now, in the $x^{(3)},y^{(3)},z^{(3)}$ coordinate system the unit vector normal to the collector plane has the components (1,0,0). Therefore by taking the scalar product of this unit vector with the one whose components are given by eq.(**2.14**) we obtain the cosine of the angle of incidence to the collector plane, i.e.

$$\cos\theta = n_{s,x}^{(3)} \qquad \textbf{(2.15)}$$

By combining eqs.(**2.15**), (**2.14**), (**2.13**), (**2.12**) and (**2.11**) we arrive at the first entry given in **Table 1**.

Example 5: Calculate the solar angle of incidence on a horizontal surface.

Solution: We take the first entry in **Table 1** and set the tilt angle $\beta = 0$. All $\sin \beta$ terms then drop out, leaving:

$$\cos \theta = \cos \delta \cos \lambda \cos \omega + \sin \delta \sin \lambda \qquad (\mathbf{2.16})$$

Notice that all terms that depend on the azimuthal angle ϕ of the receiving surface have also dropped out - which is sensible, since an upward-facing surface has no azimuthal direction. Notice also that the result depends on the geographic latitude of the site, the date (n) and the time of day (ω) all of which also make sense.

Example 6: Derive an expression for the length of day (i.e. sunrise to sunset) for any date in the year.

Solution: Sunrise occurs when $\theta = 90^0$ in eq. (**2.16**), i.e. when:

$$\cos \omega = - \tan \lambda \tan \delta \qquad (\mathbf{2.17})$$

On the other hand, solar noon occurs when $\omega = 0$. Hence, the change in hour angle from sunrise to solar noon is simply $\cos^{-1}(- \tan \lambda \tan \delta)$. Therefore, the length of day must have twice this amount, or (converting to hours):

$$\text{Sunrise -> Sunset} = (24/360^0) \times 2 \cos^{-1} (-\tan \lambda \tan \delta) \text{ hrs} \qquad (\mathbf{2.18})$$

Example 7: Compute the length of day at Beersheba for the two dates June 21 and December 21.

Solution: For Beersheba, $\lambda = 31.2^0$ N. For June 21, n = 172. For December 21, n = 355. Using eq. (**2.6**), the solar declination angle on June 21 takes the value $\delta = 23.45^0$, and for December 21, $d = -23.45^0$. Therefore, using eq. (**2.18**), the length of day in Beersheba on June 21 is 14.0 hrs. Similarly, the length of day there on December 21 is 9.97 hrs.

Example 8: Our next example treats the case of a collector that tracks the sun about a single fixed axis. Two examples of this geometry, which may be found at Sede Boqer, are a flat-plate photovoltaic system and a parabolic trough solar thermal system, both of which track the sun about fixed, horizontal, north-south axes. Our task is to find the appropriate algorithm that will enable the system's steering computer to ensure maximal reception of solar energy all day.

For this situation eq.(**2.12**) is a useful starting point since it it gives the components of the earth-sun unit vector relative to a set of axes appropriate to the problem in hand. Specifically, the $x^{(1)}$ axis points upwards, the $y^{(1)}$ axis eastwards and the north-pointing $z^{(1)}$ axis coincides with the collector's axis of rotation. See **Fig. 3**.

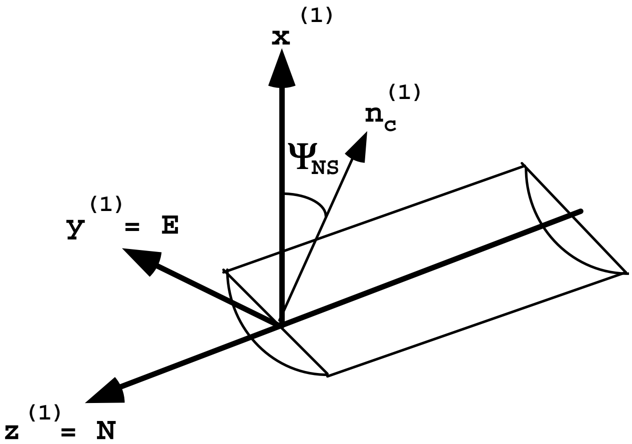

Figure 3: Appropriate Cartesian coordinate system for a NS-axis trough-like collector

We now define a tracking angle Ψ_{NS}, in such a manner that the components of the unit vector normal to the collector aperture (referred to the $x^{(1)}$,$y^{(1)}$,$z^{(1)}$ coordinate system) are:

$$\hat{\mathbf{n}}_c^{(1)} = (\cos\Psi_{NS}, -\sin\Psi_{NS}, 0) \qquad \textbf{(2.19)}$$

Note that the tracking angle is defined such that at noon $\Psi_{NS} = 0$, and in the morning it is negative.

Clearly, if the collector is to remain focused on the sun, this normal vector must be kept pointing in the direction of the projection of the earth-sun vector on a plane normal to the tracking axis of the collector. This condition may be stated mathematically as:

$$\frac{n_{c,y}^{(1)}}{n_{c,x}^{(1)}} = \frac{n_{s,y}^{(1)}}{n_{s,x}^{(1)}} \qquad \textbf{(2.20)}$$

Thus, using eqs. (**2.11**), (**2.12**) and (**2.20**), we obtain the tracking angle is the form:

$$\tan\Psi_{NS} = \frac{\sin\omega}{\cos\omega\,\cos\lambda + \tan\delta\,\sin\lambda} \qquad \textbf{(2.21)}$$

The angle of incidence θ_{NS} is then obtained by forming the scalar product of the two unit vectors (**2.12**) and (**2.19**):

$$\cos\theta_{NS} = \cos\Psi_{NS}\,(\cos\delta\,\cos\omega\,\cos\lambda + \sin\delta\,\sin\lambda) + \sin\Psi_{NS}\cos\delta\,\sin\omega \qquad \textbf{(2.22)}$$

Elimination of the tracking angle Ψ_{NS} between eqs. (**2.21**) and (**2.22**) then gives the result listed in **Table 1**.

It is evident that eq. (**2.21**) provides the necessary algorithm for a stepping motor to drive an east -> west sun-tracking solar collector. The declination angle δ changes according to the calendar, and the hour angle ω according to the clock. In actual fact the angle δ varies continuously and the angle ω undergoes a discontinuity at midnight (owing to the day length not being, in general, precisely 24 hours) thus, for high-accuracy tracking, an optical sensor would be employed in the circuitry in order to make small corrections.

REFERENCES

[**1**] **Lamm, L.O.**, *A new analytic expression for the equation of time*, Solar Energy, **26** (1981) p 465.

[**2**] **Rabl, A.**, Active Solar Collectors and Their Applications, (Oxford Univ Press, New York etc, 1985)

APPENDIX

An eleven-term Fourier fit to almanac data for the equation of time has been published by Lamm [**1**]. The following elementary algorithm (in BASIC) enables Lamm's formula to be used when eq. (**2.2**) is not accurate enough. (n = julian day number; EOT, the value of the Equation of Time, is in minutes. Claimed accuracy is 1-2 sec.). It will be observed that the four largest terms in Lamm's series are those involving cos(c), cos(2c), sin(c) and sin(2c). If these are re-arranged eq. (**2.2**) results. Thus, the remaining terms in Lamm's series are a correction to the equation given in the text above.

```
PROGRAM LAMM
10 PRINT "Enter n"
15 INPUT n
20 a0 = 2.087*10^-4
21 a1 = 9.2869*10^-3
22 a2 = -5.2258*10^-2
23 a3 = -1.3077*10^-3
24 a4 = -2.1867*10^-3
25 a5 = -1.51*10^-4
31 b1 = -1.2229*10^-1
32 b2 = -1.5698*10^-1
33 b3 = -5.1602*10^-3
34 b4 = -2.9823*10^-3
35 b5 = -2.3463*10^-4
40 c = 2*3.1415926#*n/365.25
45 d = a0+a1*COS(c)+a2*COS(2*c)+a3*COS(3*c)+a4*COS(4*c)+a5*COS(5*c)
50 e = b1*SIN(c)+b2*SIN(2*c)+b3*SIN(3*c)+b4*SIN(4*c)+b5*SIN(5*c)
55 lamm = d+e
60 EOT = 60*lamm
65 PRINT n
70 PRINT EOT
75 GOTO 10
```

HOMEWORK PROBLEMS:

1. Write an interactive computer program called "Solar angles" which incorporates Lamm's algorithm for the Equation of Time, and the relevant equations from this lecture. The program should have built in the latitude and longitude of Sede Boqer (30.9^0 N, 34.8^0 E), with a remark that will enable you to change these default coordinates to those of some other place. When run, the program should ask you to input date, time, collector azimuth angle and collector tilt angle. The program should then output the following information:

Place = Sede Boqer
Date = (input information)
Time: = (ditto)
Collector tilt = (ditto)
Collector azimuth = (ditto)
Solar noon = ??? hr
Sunrise = ??? hr
Sunset = ??? hr
Solar azimuth angle = ???
Solar zenith angle = ???
Angle of incidence on the collector plane = ???

[To check your program for errors: on Jan 21, solar noon should be at 11:52:02 and the solar zenith angle at that time = 50.9^0. Checks that the solar collector part of the algorithm is correct should consider horizontal surfaces and vertical surfaces at sunrise, solar noon and sunset].

2. Use your program to discover for Sede Boqer:
(a) The longest and shortest days of the year;
(b) The earliest time of sunrise and the date on which it occurs;
(c) The latest time of sunset and the date on which it occurs.
(d) The time of solar noon on the 21st of each month.

3. A solar collector is attached to a rotating vertical axis. The normal to its collection plane is horizontal. Compute a formula for the tracking angle that optimizes the energy collection.

4. The single-axis sun-tracking collector, described in the previous problem, is to be improved by mounting it on a rotatable horizontal axis which is rigidly attached to the vertical axis. In this manner, it becomes a dual-axis sun-tracking collector. Compute the formula that gives the required tracking angle for the collector's optimal motion about the added horizontal axis.