Rings and Coulomb boxes in dissipative environments

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We study a particle on a ring in the presence of a dissipative Caldeira-Leggett environment and derive its response to a dc field. We show how this non-equilibrium response is related to a flux-averaged equilibrium response. We find, through a two-loop renormalization group analysis, that a large dissipation parameter \( \eta \) flows to a fixed point \( \eta^\ast = \hbar/(2\pi) \). We also reexamine the mapping of this problem to that of the Coulomb box and show that the relaxation resistance, of recent interest, is quantized for large \( \eta \). For finite \( \eta > \eta^\ast \) we find that a certain average of the relaxation resistance is quantized. We propose a Coulomb-box experiment to measure a quantized noise.

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I. INTRODUCTION

Two of the most important mesoscopic structures are rings, for the study of persistent currents, and quantum dots or boxes, for the study of charge quantization. Of particular recent interest is the quantization of the relaxation resistance, defined via an ac capacitance of a single-electron box (SEB). A SEB is defined as a quantum dot that has \( N_c \) transmission channels into a single-electron reservoir (i.e., an electrode) and is capacitively coupled to a gate voltage. This setup is equivalent to an RC circuit whose capacitance at low frequency \( \omega \) has the form \( C_0(1+i\omega C_0 R_q) \), identifying the relaxation resistance \( R_q \). Following the prediction of Büttiker, Thomas, and Prêtre\(^1\) that \( R_q = \hbar/(2e^2) \) for a single channel, a quantum mesoscopic RC circuit has been implemented in a two-dimensional electron gas and \( R_q = \hbar/(2e^2) \) has been measured. The theory has been recently extended to include Coulomb blockade effects,\(^3,4\) showing that \( R_q = \hbar/(2e^2) \) is valid for small dots and crosses over to \( R_q = \hbar/e^2 \) for large dots.

In parallel, recent data has shown Aharonov-Bohm oscillations from single electron states in semiconducting rings.\(^5\) Further theoretical works have considered the effects of dissipative environments on a single particle in a ring,\(^5\) in particular studying the renormalization of the mass \( M^* \) and its possible relation to dephasing.\(^6-9\) A related case of a ring coupled by tunneling to an electron lead has also been studied.\(^10\)

It is rather remarkable that the ring and box problems are related via the Ambegaokar, Eckern, and Schön (AES) mapping\(^11\) where the ring experiences a Caldeira-Leggett (CL)\(^12\) environment. While the exact mapping assumes weak tunneling into the box with many channels, it has been extensively used to describe various tunnel junctions,\(^13\) the Coulomb blockade phenomena in SEBs, and in the single electron transistor (SET).\(^15-22\)

The ring problem is defined by a particle confined to a ring, coupled to a dissipative environment of the Caldeira-Leggett type, and in the presence of a field \( E \), generated by a time dependent flux \( \phi_E \) through the ring. This scenario is schematically illustrated in Fig. 1. The Caldeira-Leggett coupling can be realized, for example, by a normal metal whose mean-free path is much larger than the ring’s radius.\(^9\)

In the present work we address the ring problem by the real time Keldysh method and study it using a two-loop expansion and renormalization group (RG) reasoning. We find that perturbation theory identifies an unexpected small parameter \( \sin[\hbar/(2\eta)] \), where \( \eta \) is the dissipation parameter on the ring, or the lead-dot coupling in the SEB. We infer that a large \( \eta \) flows to a fixed point \( \eta^\ast \) with \( \hbar/(2\eta^\ast) = \pi \). While the thermodynamics of the ring-type problem has been much studied, including extensive Monte Carlo studies of \( M^* \), no sign of a finite-coupling fixed point has been detected. Our method evaluates the response to a strictly dc electric field \( E \), equivalent to a magnetic flux through the ring that increases linearly with time; hence a nonequilibrium response. We claim that thermodynamic quantities like \( M^* \), that are flux sensitive decouple from the response to \( E \), a response that averages over flux values. This general relation between nonequilibrium and equilibrium responses is given by Eq. (39) below. This relation has been noticed for a model with particle tunneling between a ring and an environment.\(^23\)

In terms of the SEB, our results extend the previous analysis\(^3,4\) to the case of many channels \( N_c \), an experimentally realizable scenario.\(^24\) We note that for \( N_c > 1 \) the relaxation resistance for noninteracting electrons\(^1\) becomes \( \hbar/(2N_c e^2) \). We find that for strong coupling, \( \eta/h \gtrsim 1 \), the relaxation resistance is quantized to \( e^2/h \) up to an exponentially small correction \( \sim \eta^\ast/h \). For finite \( \eta \), but still \( \eta > \eta^\ast \) we find that a certain average of the relaxation resistance is quantized [see Eq. (82)].

The present work considerably expands on our previous letter.\(^25\) In Sec. II we present the ring and box models, with some exact general properties. In Sec. III we present RG and numerical solutions for the semiclassical case, while Sec. IV presents the perturbation and RG analysis of the full quantum case. The discussion in Sec. V summarizes our results, discusses its topological interpretation, and details a proposed Coulomb-box experiment to detect our predicted quantized noise. The appendices give details of the ring-box mapping and of the various perturbation expansions. We consider temperature \( T = 0 \) throughout.

As a simple motivation for our main result, we present here a topological interpretation of the fixed point \( \eta^\ast \), based on the Thouless charge pump concept.\(^26\) Consider a slow
change of $\phi_t$ by one unit with $h\dot{\phi}_t = \eta^R(\dot{\theta})$. For the special value $\eta^R = h/(2\pi)$ the total change in the position of the particle is $\int_0^t (\dot{\theta}) dt = 2\pi t$; that is, the particle comes back to the same position on the ring and a unit charge is transported.

II. MODEL AND GENERAL PROPERTIES

A. Semiclassical model

We derive first a Langevin equation for a particle on a ring. Consider the standard Langevin equation for a particle with coordinate $x_t$ in one dimension of the form

$$R_{t,t'}x_{t'} = \xi_t,$$  

(1)

where $\xi_t$ is a Gaussian random force from an environment, where the average on the environment degrees of freedom is

$$\langle \xi_t\xi_{t'} \rangle = B_{t,t'},$$  

(2)

This relation defines a linear response for either $x_0 = R_0 \xi_0$ or $\xi_0 = R_0^{-1}x_0$, after Fourier transforms [e.g., $R_0$ is the Fourier transform of $R = R_{0,0}$]. Hence the fluctuation dissipation theorem (FDT) at temperature $T$ can be applied either way, leading to

$$K_s(\omega) = h\coth\left(\frac{1}{2}h\omega\right)\text{Im}[R_0],$$  

(3)

$$B_\omega = h\coth\left(\frac{1}{2}h\omega\right)\text{Im}[R_0],$$  

where $K_s(\omega)$ is the Fourier transform of $K_s(\tau) = \frac{1}{2}(x_{t} + x_{t+t'})$. The simplest choice corresponds to a particle with mass $m$ and friction coefficient $\eta$, so that at temperature $T = 0$,

$$m\ddot{x}_t + \eta \dot{x}_t = \xi_t,$$  

$$R_0(\omega) = \frac{-1}{\omega^2 + i\omega\eta}, \quad R_0(t) = \frac{1}{\eta}[1 - e^{-\eta t/m}]\Theta(t),$$  

$$B_\omega = \frac{\hbar\eta}{\pi t^2}, \quad B_t = \frac{-\hbar\eta}{\pi t^2} \quad (t \neq 0),$$  

(4)

where $\Theta(t)$ is the Heaviside function and $R_0(t - t')$ is the response in this case. While the mass provides a high-frequency cutoff which we denote as $\omega_0 = \eta/m$, the singularity of $B_t$ at $t = 0$ implies the need for an additional cutoff. This additional cutoff is a convenience and will be used below in the simulations as well as in the RG derivation.

A method for deriving general response functions is based on Kramers-Kronig relations.\textsuperscript{27} In the notation of Eq. (2.7) of Ref. 27 we choose $\text{Re} \mu(\omega) = \eta/(1 + \omega^2\tau_0^2)$ so that the response function $R_{0,-t'}^{-1}$, after Fourier transform, is

$$R_{-t'}^{-1} = -m\omega^2 - \frac{i\omega\eta}{1 - i\omega\tau_0}. \quad (5)$$

To justify the use of this form it suffices to say that it has the remarkable and necessary property that both $R_0$ and $R_{-t'}^{-1}$ have no poles in the upper half plane, as needed for causal functions; [note that $R_0$ reduces to $R_0(\omega)$ when $t_0 = 0$]. The FDT at $T = 0$ gives

$$B_\omega = \frac{\hbar|\omega|\eta}{1 + \omega^2\tau_0^2},$$  

(6)

so that $1/\tau_0$ provides a cutoff for the environment frequencies, in addition to the cutoff $m/\eta = 1/\omega_0$. Hence for $4\tau_0 < m/\eta (\delta \rightarrow +0)$,

$$R_t = \Theta(t)\frac{\tau_0}{m}\left[\frac{m}{\eta t_0}e^{-\delta t} - \frac{1 - \lambda_1}{\lambda_1 x}e^{-\lambda_1 t/m} - \frac{1 - \lambda_2}{\lambda_2 x}e^{-\lambda_2 t/m}\right],$$  

$$\lambda_1 = \frac{1}{2}[1 + x], \quad \lambda_2 = \frac{1}{2}[1 - x], \quad x = \sqrt{1 - 4\eta\tau_0/m},$$  

(7)

while for $4\tau_0 > m/\eta$ with $x = \sqrt{4\eta\tau_0/m - 1}$,

$$R_t = \Theta(t)\left[\frac{1}{\eta}e^{-\delta t} - \frac{1 - x^2}{2x} \sin(x t/2\tau_0) + \cos(x t/2\tau_0)e^{-t/2\tau_0}\right].$$  

(8)

Consider now the two-dimensional system and its projection on a ring [i.e., $x_0 = (\cos\theta_t, \sin\theta_t)$], so that $\theta_t$ is the angular position of the particle and the radius is chosen as unity. In cartesian coordinates we define random forces in the $x, y$ directions so that $R_{1,-t'}^{-1}\cos\theta_t = -\xi_t^a, R_{1,-t'}^{-1}\sin\theta_t = \xi_t^b$. The ring potential confines the motion to the azimuthal part, so that only the tangent force $-\xi_t^a\cos\theta_t + \xi_t^b\sin\theta_t$ is allowed, hence

$$-\sin(\theta_t)R_{1,-t'}^{-1}\cos\theta_t + \cos(\theta_t)R_{1,-t'}^{-1}\sin\theta_t = \xi_t^a\cos\theta_t + \xi_t^b\sin\theta_t + E,$$  

(9)

where $\xi_t^a, \xi_t^b$ are independent and each have the correlations of Eq. (2). An external tangent electric field $E$ has been added corresponding to a flux through the ring that is increasing linearly with time $\phi_t = Et$. With $R_0(t - t')$ given by Eq. (4) the differential form $R_0^{-1}(t) = m\ddot{\theta}_t + \eta\dot{\theta}_t$ can be used leading to

$$m\ddot{\theta}_t + \eta\dot{\theta}_t = \xi_t^a\cos\theta_t + \xi_t^b\sin\theta_t + E.$$  

(10)
equivalent to a partition function

\[ Z = \int \mathcal{D}[\theta, \xi] \delta(m\dot{\theta} + \eta \theta - \xi^a \cos \theta - \xi^b \sin \theta - E) \]

\[ \times \exp \left\{ - \int_0^\infty \left[ \frac{|\xi^a|^2}{\beta^2} + \frac{|\xi^b|^2}{\beta^2} \right] / (2B\omega) \right\}. \]  

(11)

Introducing the “quantum” field \( \hat{\theta} \) by \( \delta(X_t) = \int \mathcal{D}[\hat{\theta}] e^{i\hat{\theta}X_0} \), and averaging over the noise field \( \xi, \xi_y \) results in the semiclassical partition function \( Z = \int \mathcal{D}[\hat{\theta}] e^{-S[\theta, \dot{\theta}]} \) where \( S[\theta, \dot{\theta}] = S_0 + S_{\text{int}} \) is given by the \( t, t' \) integrations

\[ S_0 = i \int_{t,t'} \dot{\theta}_i R^{-1}_{i,t'} - i E \int_t^{t'} \dot{\theta}_i, \]

\[ S_{\text{int}} = \frac{1}{2} \int_{t,t'} \dot{\theta}_i B_{i,t'} \dot{\theta}_i \cos(\theta_i - \theta_{t'}). \]  

(12)

This has the form of a Keldysh action, with \( \theta, \dot{\theta} \) being the classical and quantum fields, respectively. We will see below that this action is the semiclassical \( \hbar \to 0 \) limit of the full quantum system.

**B. Quantum model**

We proceed to define the full quantum problem. The one-dimensional Langevin system has the Keldysh partition \( Z = \int \mathcal{D}[\xi^a, \xi^b] e^{-S_K} \) where

\[ S_K = i \int_{t,t'} x_i R^{-1}_{i,t'} x_i + \frac{1}{2} \int_{t,t'} \dot{x}_i B_{i,t'} \dot{x}_i \]  

(13)

and \( x_i, x_t \) are the quantum and classical fields, respectively,

\[ x_i = \frac{1}{2} (x_i^+ + x_i^-), \quad \dot{x}_i = \frac{1}{\hbar} (x_i^+ - x_i^-), \]  

(14)

and \( x_i^\pm \) are on the upper and lower Keldysh contour, respectively. On a ring, we use a two-dimensional vector notation

\[ x_i^+ = [\cos \theta_i^+, \sin \theta_i^+], \quad x_i^- = [\cos \theta_i^-, \sin \theta_i^-]. \]  

(15)

Defining

\[ \theta_i = \frac{1}{2} (\theta_i^+ + \theta_i^-), \quad \dot{\theta}_i = \frac{1}{\hbar} (\theta_i^+ - \theta_i^-), \]  

(16)

and using trigonometric identities we obtain the quantum action

\[ S_K = \frac{i}{\hbar} \int_{t,t'} R_{i,t'}^{-1} \sin \left( \frac{\hbar}{2} \dot{\theta}_i \right) \cos \left( \frac{\hbar}{2} \dot{\theta}_i \right) \sin(\theta_i - \theta_t) \]

\[ + \frac{2}{\hbar^2} \int_{t,t'} B_{i,t'} \sin \left( \frac{\hbar}{2} \dot{\theta}_i \right) \sin \left( \frac{\hbar}{2} \dot{\theta}_i \right) \cos(\theta_i - \theta_t). \]  

(17)

We note that the path integral involves continuous \( \theta \) trajectories that can involve \( n \) rotations around the ring. Consider the time evolution from an initial wave function \( \psi(\theta_0, t_0) \) at time \( t_0 \) to a final state \( \psi(\theta_t, t) \), where both initial and final angles are compact, \( 0 < \theta_0, \dot{\theta}_t < 2\pi \),

\[ \psi(\theta_t, t) = \int_0^{2\pi} d\theta_0 \sum_n \int_{\theta_0}^{\theta_0 + 2\pi n} D\theta e^{-S(\theta, \dot{\theta})} \psi(\theta_0, t_0). \]  

(18)

The sum on the integers \( n \) expresses the probability to arrive at a given \( \dot{\theta}_t \) is a sum of probabilities, each with \( n \) rotations. The path integral can therefore be written in terms of a decompactified variable \( \theta_t = \theta_t + 2\pi n \) (i.e., \( \sum_n \int_{\theta_0}^{\theta_0 + 2\pi n} D\theta \to \int_{\theta_0}^{\theta_0 + 2\pi} D\theta \) where now \( -\infty < \theta_t < \infty \)). This shift does not affect the periodic terms in Eq. (17); however, it does affect an external electric field \( E \). Consider a time-dependent flux \( \phi_i(t) = E t \) that contributes to the action a term

\[ \int_{t_t}^{t'} \phi_i(t) \dot{\theta}_i dt = -E \int_{t_t}^{t'} \dot{\theta}_i dt + \phi_i(t) \dot{\theta}_i - \phi_i(t) \dot{\theta}_i. \]

The partial integration is allowed only for the decompactified variable \( \theta_t \) (i.e., the work done by \( E \) is finite for each \( 2\pi \) rotation). The boundary terms are neglected; for example, one can choose \( \phi_i(t_0) = \phi_i(t_f) = 0 \) where \( t_0, t_f \to -\infty \) are boundary times on a Keldysh contour; the field \( E \) is turned on slowly away from these times.

In the following we will consider a perturbative scheme with a field \( E \) and a bare velocity \( v = E/\eta \) and with \( \theta_t \) decomposed to \( \theta_t = \delta \theta_t + v \eta \) [the true velocity is defined below as \( v^2(E) = \langle \dot{\theta}_i \rangle \)]. The velocity \( v \) provides a low-frequency cutoff eliminating divergence of the perturbative expansion and eventually allows for RG treatment. It will be convenient to use the two-cutoff response \( R^{-1}_\omega = -m \omega^2 + \delta R^{-1}_\omega \), where \( \delta R^{-1}_\omega = -i\eta v/(1-i\tau\omega) \), hence

\[ \delta R^{-1}_{\omega, \tau'} = -\eta \int_0^{\tau'} \frac{-\eta \omega}{1 - i\omega \tau} e^{-i\omega(t-t')} \]

\[ = -\eta \frac{\tau \theta_t}{\tau_0} e^{-i\omega(t-t')} \Theta(t-t'). \]  

(19)

The operator identity is satisfied for any function decaying faster than \( e^{-r/\tau_0} \) at \( t' \to -\infty \). Note,

\[ i \int_{t,t'} \dot{\theta}_i R^{-1}_{\omega, \tau'} \dot{\theta}_i = i \int_0^{\tau} \dot{\theta}_i \int_{-\infty}^{t'} e^{-i\omega(t-t')/\tau} dt' = i \eta \int_0^{\tau} \dot{\theta}_i. \]  

(20)

The mass term with \( m \omega^2 \to \delta(t-t') \partial_t \partial_{t'} \) produces \( m \int_0^{\tau} \dot{\theta}_t \dot{\theta}_t = m \int_0^{\tau} \dot{\theta}_t \dot{\theta}_t + m^2 \int_0^{\tau} \dot{\theta}_t \dot{\theta}_t \); the last term with \( m^2 v = E/\omega_0 \) is neglected relative to the field term \( \int_0^{\tau} E t \dot{\theta}_t \). The full action is then

\[ S_K = S_0 + S_{\text{int}} + S_\tau, \]

\[ S_0 = i \int_{t,t'} \dot{\theta}_i R^{-1}_{\omega, \tau'} \dot{\theta}_i - i E \int_{t,t'} \dot{\theta}_i = i \int_{t,t'} \dot{\theta}_i R^{-1}_{\omega, \tau'} \dot{\theta}_i, \]

\[ S_{\text{int}} = \frac{2}{\hbar^2} \int_{t,t'} B_{i,t'} \sin \left( \frac{\hbar}{2} \dot{\theta}_i \right) \sin \left( \frac{\hbar}{2} \dot{\theta}_i \right) \cos(\theta_i - \theta_i), \]  

(21)

\[ S_\tau = i \int_{t,t'} \delta R^{-1}_{\omega, \tau'} \]  

\[ \times \left[ \sin \left( \frac{\hbar}{2} \dot{\theta}_i \right) \cos \left( \frac{\hbar}{2} \dot{\theta}_i \right) \sin(\theta_i - \theta_i) - \frac{\hbar}{2} \dot{\theta}_i \right]. \]
The use of a single cutoff \((4)\) with
\[
R_0^{-1}(t,t') = \delta(t-t') [m \dot{\theta}_t \partial_t + \eta \partial_t \dot{\theta}_t]
\] (22)
leads to a simpler action. It corresponds to \(\tau_0 \to 0\), hence \(\delta R_{t,t'}^{-1} \to \eta \delta(t-t') \partial_t\).
\[
\frac{2}{\hbar} R_0^{-1}(t,t') \sin \frac{\hbar}{2} \dot{\theta}_t \cos \frac{\hbar}{2} \dot{\theta}_t \sin(\theta_t - \theta_t') = \delta(t-t') \left[ m \dot{\theta}_t \partial_t + \frac{\eta}{\hbar} \sin(\hbar \dot{\theta}_t) \partial_{\dot{\theta}_t} \right],
\] (23)

where \(t^-\) is infinitesimally below \(t\) so that the retarded nature of \(R_{t,t'}^{-1}\) is maintained. The action \(S_K = S_0 + S_{\text{int}} + S_c\) is then
\[
S_0 = i \int_t (m \dot{\theta}_t \partial_t + \eta \dot{\theta}_t \partial_{\dot{\theta}_t}) dt = i \int_t \left[ m \dot{\theta}_t \partial_t + \eta \dot{\theta}_t \partial_{\dot{\theta}_t} \right] dt,
\] (24)

Note that this action reduces to that of the semiclassical case \((12)\) when \(\hbar \to 0\).

### C. Renormalized friction

The renormalized friction \(\eta^R(E)\) is defined by the renormalized response \(R_{t,t'}^R = i \langle \dot{\theta}_t \dot{\theta}_t \rangle_E\) and its dc limit:
\[
\frac{1}{\eta^R(E)} = \lim_{\omega \to 0} \left( -i \omega R_0^R \right),
\] (25)
in analogy with the bare form \((4)\). We show now that the renormalized \(\eta^R(E)\) is also the local slope of \(d v^R / d E\), where \(v^R\) is the \(E\)-dependent renormalized velocity
\[
v^R \equiv \langle \dot{\theta}_t \rangle = \int D[\theta] \dot{\theta}_t e^{-S_K}.\]
(26)

Therefore,
\[
d v^R / dE = i \int_t \left[ m \dot{\theta}_t \partial_t + \eta \dot{\theta}_t \partial_{\dot{\theta}_t} \right] dt = \lim_{\omega \to 0} \left[ -i \omega \right] R_0^R e^{-i \omega t} = \frac{1}{\eta^R(E)}.\]
(27)

In particular we are interested in the limit \(E^R = \eta^R(E \to 0)\).

We show now an alternative procedure for evaluating \(\eta^R\). Consider the Keldysh partition \(Z = \int D[\theta] e^{-S_K}\) and shift \(\dot{\theta}_t \to \dot{\theta}_t + a_t\). The result must be \(a_t\) independent, and by choosing the form \((23)\) with \(\tau_0 \to 0\) (the following identity is actually independent of cutoff choices),
\[
0 = \frac{\delta Z}{\delta a_t} \bigg|_{t=0} = \left\langle \delta(S_0 + S_{\text{int}} + S_c) \right\rangle_{\theta_t} = -i(\eta v^R - E - \delta E),
\] (28)
since \(-i(\delta S_0 / \delta \dot{\theta}_t) = -m(\dot{\theta}_t) + \eta(\dot{\theta}_t) - E\) and \(v^R\) is time independent, at least for long times.

Taking an \(E\) derivative of Eq. \((28)\) and using Eq. \((27)\) we obtain
\[
\frac{1}{\eta^R(E)} = \frac{1}{\eta} + \frac{1}{\eta^R} \frac{\partial}{\partial E} \delta(E).
\] (29)

We have checked, up to second-order terms, that the results of Eqs. \((27)\) and \((29)\) coincide. The use of Eq. \((29)\) is technically easier.

### D. Equilibrium correlations

In this section we consider the equilibrium response to a change in flux and derive a relation with the nonequilibrium response to a field.

Consider now the form of \(\tilde{K}(\omega)\) as a response to a flux \(\phi\). Linear response to \(\delta H_{\text{tang}} = -\hbar \dot{\theta}_t \delta \phi_t(t)\) is
\[
\dot{\hbar}(\theta) = -\int \tilde{K}_{t,t'} \delta \phi_t(t').\]
(30)

This corresponds also to the velocity correlation
\[
\tilde{K}_{t,t'} = +i \theta(t-t') \langle [\dot{\theta}_t, \dot{\theta}_{t'}] \rangle.
\] (31)

We expect that the dc response is positive for small \(\phi\), so we define
\[
\tilde{K}(\omega) = -K_0(\phi) + i \omega K_1(\phi) + O(\omega^3).
\] (32)

The response \(K_0(\phi)\) is the persistent current; that is, for a static flux one can integrate Eq. \((30)\):
\[
\langle \dot{\theta} \rangle = \int_0^{\phi} K_0(\phi') d\phi'.
\] (33)

The periodicity of the persistent current implies \(\int_0^1 K_0(\phi) d\phi = 0\). The curvature of the free energy \(F\) (or energy at \(T = 0\)) at \(\phi = 0\) is a well-studied object.\(^6–^9\)

For general \(\phi\), it is defined by a Matsubara imaginary time connected correlation
\[
\langle [\dot{\theta}_t, \dot{\theta}_{t'}] \rangle = (\beta)^{-1} \int_0^\beta \int_0^\beta \dot{\theta}_t \dot{\theta}_{t'} e^{i \tau \tau} d\tau d\tau' = K_0(\phi),\]
(34)

where \(K(\omega_n = 0) = +K_0\) (there is a sign difference in the standard Matsubara notation). An effective mass is defined by \(K(0) = \hbar / M^*\) so that \(M^* = m\) without interactions, while for strong \(\eta \gg 1\) coupling \(M^* \sim e^{\eta/2}\) is exponentially large.\(^6–^9\)

To appreciate the role of \(K_1\) consider FDT for the symmetrized correlation at small \(\omega\)
\[
\langle [\phi, \hat{\phi}] \rangle_{\phi = \omega} = \text{sgn}(\omega) \text{Im} \tilde{K}_{\omega} = |\omega| K_1.
\] (35)

The diffusion involves the response \(\langle [\phi, \hat{\phi}] \rangle = K_1 / |\omega|\), hence for \(\tau \to \infty\)
\[
\langle (\phi - \bar{\phi})^2 \rangle = K_1 \int d\omega \frac{1 - \cos \omega t}{\pi |\omega|} = \frac{2K_1}{\pi} \ln(\omega_0 t),
\] (36)

where \(\omega_0\) is a characteristic frequency where higher-order terms in \(\omega\) terms set in.
Consider now the linear response to an electric field $\delta H_{\text{rad}} = -E(t)\hat{\theta}$, and use the response $\langle \hat{\theta} \rangle = R^R E(t')$. The definition (25) implies that the low-$\omega$ limit has the form $R^R_\omega = -1/(i\omega R_q)$. Since $E = \hbar \dot{\theta}$, we expect $\hbar \omega^2 R^R_\omega = \tilde{K}(\omega)$. However, there is a difficulty with the latter relation, if taken literally,
\begin{equation}
\frac{-\hbar \omega^2}{i\omega R_q} = \frac{-1}{\hbar} K_0(\phi_x) + i\omega K_1(\phi_x).
\end{equation}
It is also not clear which $\phi_x$ to use in this relation. To resolve this issue consider the $\tilde{K}$ response with a constant electric field
\begin{equation}
\hbar \langle \dot{\theta} \rangle = -\int E(t') \tilde{K}_{1,t'} dt'.
\end{equation}
Note first that an additional constant $\phi_x$ in $E(t')/\hbar + \phi_x$ can be eliminated by redefining the origin of the time $t'$, hence the persistent current part should be eliminated. More precisely, define $\phi_x(t) = E(t)/\hbar$; the $\omega = 0$ component $K_0(\phi_x) = K_0(\tilde{E}/\hbar)$ is a periodic function [i.e., an ac response with frequency $\omega_0 = (2\pi/\hbar)E$]. For $\omega \to 0$ this persistent current response averages to zero (i.e., $\int_0^1 K_0(\phi_x) d\phi_x = 0$). The same reasoning applies to a $\phi_x$ average on $K_1(\phi_x)$. Hence for the purpose of evaluating the dc response of Eq. (25) we need to average on the flux in Eq. (32), hence
\begin{equation}
\lim_{E \to 0} \lim_{\omega \to 0} \frac{\tilde{K}(\omega)}{i\omega} = \frac{1}{\hbar} \int_0^1 K_1(\phi_x) d\phi_x = \frac{\hbar}{\eta R_q}.
\end{equation}
The order of limits in Eq. (5) signifies that $\eta^R$ is essentially a nonequilibrium response. The equilibrium–nonequilibrium relation (39) has been noticed in the solution of a Boltzmann relaxation equation for particles on a ring, allowing for particle tunneling into an environment.\(^2\)

The physical picture is that in a dc field the particle rotates around the ring and produces two types of currents. First is the persistent current that oscillates in time as $\phi_x$ increases and therefore time averaged to zero; this current is nondissipative. Second, there is a genuine dc response from the $i\omega K_1$ term, which is dissipative.

### E. Coulomb box

Consider now the Coulomb-box system; namely, a finite region (a “dot”) with charging energy $E_c$ coupled by tunneling to a single metallic lead. The Hamiltonian is
\begin{equation}
\mathcal{H} = \sum_k \epsilon_k a_k^\dagger a_k + \sum_{a,i} \epsilon_a d_{a,i}^\dagger d_{a,i} + E_c (\hat{N} - N_0)^2 + \sum_{k,a,i} t_{k,a} a_k^\dagger d_{a,i} + \text{H.c.},
\end{equation}
where $i = 1, \ldots, N_c$ are channel indices, $d_{a,i}$ are dot electron operators with spectra $\epsilon_a$, $a_k$ are lead electron operators with spectra $\epsilon_k$, $\hat{N} = \sum_{a,i} d_{a,i}^\dagger d_{a,i}$ is the number operator on the dot, $E_c = e^2/(2C_g)$ is the charging energy with $C_g$ is the geometric (bare) capacitance, and $N_0$ is the gate voltage in units of $2E_c$. The channel index $i$ is diagonal in the tunneling term (i.e., corresponds to transverse modes that are conserved in tunneling).

Consider the density correlations
\begin{equation}
K_{t,t'} = +i\theta(t - t')[\hat{N}_t, \hat{N}_{t'}].
\end{equation}
The AES mapping to the ring problem is reproduced in Appendix A. In particular, $N_0$ corresponds to $-\phi_x$, $2E_c$ to $\hbar^2/m$, and the relation to the velocity correlation on the ring is
\begin{equation}
\hbar^2 \tilde{K}_{t,t'} = -2E_c \hbar \delta(t - t') + 4E_c^2 K_{t,t'}.
\end{equation}
Using the notation\(^3\) $K(\omega) = \hbar C_0(1 + i\omega C_0 R_q)^{-1/2}$, where $C_0$ is the renormalized capacitance and $R_q$ is the relaxation resistance, we obtain
\begin{equation}
\hbar \tilde{K}(\omega) = -2E_c + 4E_c^2 \frac{C_0}{C_g} (1 + i\omega C_0 R_q).
\end{equation}
Hence the mapping between the Coulomb box and the ring for the curvature is, using Eq. (34),
\begin{equation}
\frac{\hbar^2}{M^*(\phi_x)} = \hbar K_0(\phi_x) = 2E_c \left(1 - \frac{C_0}{C_g}\right) \Rightarrow \frac{m}{M^*(\phi_x)} = 1 - \frac{C_0(N_0)}{C_g},
\end{equation}
while for the dissipation, using Eq. (39),
\begin{equation}
\frac{\hbar}{\eta R_q} = \frac{1}{h} \int_0^1 K_1(\phi_x) d\phi_x = e^2 \sqrt{\frac{C_0(N_0)}{C_g}} R_q(N_0) dN_0.
\end{equation}
We note that $\int_0^1 K_0(N_0)/C_g dN_0 = 1$ due to the periodicity of $F(\phi_x)$. An extensive study\(^6-9\) of $M^*(0)$ shows that it satisfies $M^*(0) > m$ and that for large $\eta$ (the bare interaction parameter) $M^*(0)/m \sim e^{\eta \eta} \gg 1$. Hence,
\begin{equation}
\frac{C_0}{C_g} = 1 - O(e^{-\eta}), \quad \eta \gtrsim 1,
\end{equation}
and $C_0 \to C_g$ for large $\eta$.

At this stage we can already propose an interesting experiment for the SEB. By analogy with $E = \hbar \phi_x$ in the ring, we propose measuring the response to a gate voltage that is linear in time $N_0 \sim t$. This leads to a dc current into the Coulomb box whose dissipation is the average in Eq. (45). This average is predicted to be quantized, at least for $\eta > \eta^R$, as shown below.

### III. SEMICLASSICAL RENORMALIZATION GROUP AND NUMERICS

#### A. Perturbations and renormalization group
We study here the action (12) with a perturbation series for correlation functions. Consider first the correlation $C_{t,t'} = \langle \theta_t \theta_{t'} \rangle$, which to first order is
\begin{equation}
C^{(1)}_{t,t'} = \langle \theta_t \theta_{t'} \rangle - S_{\text{int}} = \int_{t_1, t_2} B_{t_2, t_1} \cos \theta(t_1 - t_2) R_{t_1, t_2} R'_{t_2, t_1}.
\end{equation}
In Fourier space
\begin{equation}
C^{(1)}_{t,t'} = |R_{t,t'}|^2 B^{(1)}_{t,t'},
\end{equation}
where $B^{(1)}_{t,t'} = \frac{1}{2}(B_{t_1, t_2} + B_{t_2, t_1})$. Since $C^{(1)}_{t,t'}$ is divergent it is useful to evaluate $\tilde{C}_{t,t'} = \langle |\theta_t - \theta_{t'}|^2 \rangle$, which to first order is,
with $\tau = t - t'$ ($\tau \gg 1/\omega_c$),

$$
\tilde{C}_t = \int_0^\infty B_{0t}^* |S_{0t}|^2 (1 - \cos \omega \tau) \\
\approx \frac{2h}{\pi \eta} \left( \frac{\ln (\frac{\eta}{\eta_r})}{\pi \eta \tau}, \quad \tau < 1/v, \quad \frac{1}{v} < \tau. \right) \tag{49}
$$

For $E = 0$ the angular position diffuses logarithmically, while for $E \neq 0$ the long-time fluctuation is linear in time.

Consider next the response function to second order in $S_{rt}$,

$$
R_{rt} = i \{ \tilde{R}_{rt} + R_{rt}^{(1)} + R_{rt}^{(2)} \}
$$

$$
= R_{rt} + i \{ \tilde{R}_{rt} + R_{rt}^{(1)} \} \left( -S_{rt} + \frac{1}{2} \delta^2 \right). \tag{50}
$$

Note that the disconnected terms in the perturbation $(S_{rt})^2$ vanish for any order $n$, due to the normalization $Z = 1$. The first-order response function is

$$
R_{rt}^{(1)} = -i \frac{1}{2} \int_{t_1, t_2} B_{t_1, t_2} [\tilde{g}_{t_1, t_2} \cos (\theta_{t_1} - \theta_{t_2})]. \tag{51}
$$

The result in the frequency variable is (see Appendix B)

$$
R_{rt}^{(1)} = R_{rt}^2 \int_{\omega_0} \omega_0 [B_{\omega_0}^* - B_{\omega_0}^v] \\\n= R_{rt}^2 \int \omega_0 B_{\cos \omega (e^{i\omega t} - 1)}. \tag{52}
$$

We note that for $\omega = 0$ FDT is maintained, to this order, $C_{t, t}^{(0)} = \int R_{rt} \Re \arg(\omega)$. The renormalized $\eta$ to first order is then

$$
\frac{1}{\eta_1^2} = \lim_{\omega \to 0} (-i\omega) R_{\omega}^{(1)} = \lim_{\omega \to 0} \frac{-i\omega}{(-i\omega)^2} \int \omega_0 B_{\omega} \cos \omega (e^{i\omega t}) \\\n= \frac{1}{2} \ln (1 + \omega^2 / \eta^2) = -\frac{\ln \omega / \omega_c}{\eta^2} + O(v). \tag{53}
$$

Considering next the second order in Eq. (50) we obtain (see Appendix B)

$$
R_{rt}^{(2)} = R_{rt}^2 \left( -\frac{1}{2} \int \omega_0 B_{\omega} \cos \omega (e^{i\omega t} - 1) C_{t, t}^{(1)} \right. \\\n+ \frac{1}{2} \int \omega_0 B_{\omega} \cos \omega (e^{i\omega t} - 1) \\\n+ R_{t, t} \left( \int \omega_0 B_{\omega} \cos \omega (e^{i\omega t} - 1) \right)^2 \\\n- \int_{t_1, t_2} R_{t_1, t_2} \sin \omega t_1 \sin \omega t_2 (1 - e^{i\omega t_1}) T_1. \right) \tag{54}
$$

Denoting the contribution of the last term in Eq. (54) as $\delta(1/\eta_2^R)$ we obtain for the renormalized dissipation to second order (with $\ln v \to \ln v / \omega_c$ implied below)

$$
\frac{1}{\eta_2^R} = \frac{1}{\eta} - \frac{\ln v / \eta^2}{\eta^2} + \frac{\ln v / \omega_c}{\eta^2} + \delta(1/\eta_2^R). \tag{55}
$$

The contribution of the last term is peculiar and depends on the order in which the limits are taken. We define a nonequilibrium limit where $\eta_2^R$ is evaluated for a strictly dc field (i.e., $\omega \to 0$ is taken first) and then a logarithmically divergent $E \neq 0$ term is obtained; namely,

$$
\delta(1/\eta_2^R) = \frac{1}{\eta^2} \lim_{v \to 0} \lim_{\omega \to 0} \frac{1}{i \omega} \int \omega_0 B_{\omega} \sin \omega (e^{i\omega t_1}) T_1 \\\n\times \frac{\ln v / \eta^2}{\eta^2} + \frac{\ln v / \omega_c}{\eta^2} + \eta^2 \int \omega_0 B_{\omega} \sin \omega (e^{i\omega t_1}) T_1 \\\n\times \left( \frac{1}{\eta^2} - \frac{1}{\eta_2^R} \right) + \frac{1}{\eta} \ln v + O(v) = \frac{1}{\eta^2} \ln v. \tag{56}
$$

Considering next the alternative equilibrium order of limits (i.e., first $E \to 0$), we obtain

$$
\lim_{\omega \to 0} \lim_{v \to 0} \frac{1}{\eta^2} \ln \sin \omega (e^{i\omega t_1}) T_1 = 0, \tag{57}
$$

hence $\delta(1/\eta_2^R) = 0$. The renormalized $\eta$ to second order is then

$$
\frac{1}{\eta_2^R} = \frac{1}{\eta} - \frac{\ln v / \eta^2}{\eta^2} + \frac{\ln v / \omega_c}{\eta^2} + \frac{\ln v}{\eta_2^R}. \tag{58}
$$

We choose the cutoff to be in Lorentzian form $B_\omega = \frac{b_0}{\eta^2} |\omega| \left( 1 + \omega^2 / \omega_c^2 \right)$, in the following section we explain the importance of this choice.

We write the equation in an iterative procedure. Using the convolution form

$$
\theta_t = \int_{t'} R_{t, t'} \left( \xi_t \cos \theta_{t'} + \xi_{t'}^2 \sin \theta_{t'} - E \right), \tag{59}
$$

starting with an arbitrary configuration of $\theta_0^{(0)}$ we calculate the right-hand side (RHS) of (59) to find a new $\theta_1^{(1)}$. We repeat the procedure $n$ times until the expression is saturated when $\theta_0^{(n)} = \theta_1^{(n+1)}$. This procedure is improved if instead of taking the convolution result as the next order $\theta_t$, we use some mixing of that result and of the previous $\theta_t$ configuration in the form $\theta_t^{(m)} = (1 - \beta) \theta_t^{(m-1)} + \beta \cdot \text{RHS}$ where $\beta$ is a mixing parameter. Typically $\eta$ would be of the order of $10^5$ and $\beta = 0.1$.

With this choice the Langevin equation takes the following form:

$$
\frac{m}{\tau_0} \ddot{\theta}_t = \xi_t \cos \theta_t + \xi_{t'}^2 \sin \theta_{t'} + E + \Delta_t, \tag{60}
$$

$$
\Delta_t = \frac{\eta}{\tau_0} \int_{-\infty}^t \sin [\theta_t - \theta_{t'}] e^{-(t-t_0)/\tau_0} dt',
$$

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where $\Delta_t$ is a correction term defined by $\delta R^{-1}_w$ in the response function (19) because $\int \delta R^{-1}_w \{ \xi_t^2 \cos \theta_t + \xi_t^2 \sin \theta_t + E \} = -\int \delta \omega \delta R^{-1}_w \Delta_\omega$.

In the numerical system we have now four time scales, two numerical time scales, i.e. the time segment $\Delta t = \bar{T}/N$ and the time span $\bar{T}$, as well as the two physical high-frequency cutoffs, $1/\tau_0$ for the noise and $\omega_c$ the mass cutoff. The region of interest corresponds to velocity $vR = \langle \dot{\theta}_t \rangle$ between the time scales $\Delta t \ll \tau_0 < 1/\omega_c \ll 1/vR \sim 1/v < \bar{T}$. The inequality $\tau_0 < 1/\omega_c$ is useful since we compare the numerical result to an asymptotic result in which $\omega_c$ rather than $1/\tau_0$ is the high-frequency cutoff.

With the result for $\dot{\theta}_t$, we can find the renormalized $1/\eta v = d\xi_R/dE$ with $vR = \langle \dot{\theta}_t \rangle$ where the average $\langle \ldots \rangle$ reflects an average on both the time domain $t > 1/\omega_c$ and on numerous realizations of the noise.

In the left panel of Fig. 2 our numerical solution for the Langevin equation is shown, including a fit to the second order with $b_0 = 0$. On the right panel the first order is subtracted with either the nonequilibrium $b_0 = 0$ or the equilibrium $b_0 = -1$. The first is in fact a better fit for the numerical data. When $1/v$ approaches the simulation time span $\bar{T}$ the numerics become unreliable, as the particle cannot complete even one revolution in time $\bar{T}$; a plateau is then observed at low $E$.

With the numerical results for $\dot{\theta}_t$ we can also generate the correlation function $\tilde{C}_\tau = \langle [\theta_t - \theta_0]^2 \rangle$, the first-order perturbation for this correlation function is given in Eq. (49). In Fig. 3 we plot this correlation function as a function of the time separation $\tau$ for the same parameters as in Fig. 2, with and without a finite field. The data is fairly close to the first-order result (49) for not too long times; that is, for zero field the correlation has a subdiffusion logarithmic behavior while for finite force the correlation has a diffusion ($\sim \tau$) behavior.
IV. QUANTUM RENORMALIZATION GROUP

A. Perturbations from $S_{\text{int}}$

Consider now the definition $\eta^R$ in Eqs. (28) and (29):

$$-i\delta E^{(1)} = \frac{\delta S_{\text{int}}}{\delta \delta_t} = \frac{2}{\hbar} \int B_{i,r} \left[ \frac{\hbar}{2} \sin \left( \frac{\hbar}{2} \delta_t \right) \cos(v t - v t' + \delta t_i - \delta t_f) \right]_0$$

$$= \frac{2}{\hbar} \int \sum_{\sigma, \sigma', \mu = \pm} \frac{\delta}{\delta t} \left[ \exp \left[ i \frac{\hbar}{2} (\sigma \delta_t + \frac{1}{2} i \mu (v t - v t' + \delta t_i - \delta t_f)) \right]_0 \right]$$

$$= \frac{2}{\hbar} \int \sum_{\sigma, \sigma', \mu = \pm} \frac{\delta}{\delta t} \left[ \exp \left[ - \frac{1}{2} \mu \delta t \right] \right] = \frac{2}{\hbar} \int \sum_{\sigma, \sigma', \mu = \pm} \frac{\delta}{\delta t} \left[ \exp \left[ - \frac{1}{2} \mu \delta t \right] \right]$$

(61)

For $t < t'$ the term $\sigma' R_{t'} = 0$ and then $\sum_{\sigma'} = 0$. The result is then finite only for $t > t'$; defining $\mu' = \mu \sigma'$,

$$= \frac{2}{\hbar} \int \sum_{\sigma, \mu = \pm} \frac{\delta}{\delta t} \left[ \exp \left[ i \sigma' \mu (v t - v t') + \frac{1}{2} \frac{\hbar}{2} \mu' R_{t'} \right] \right] = \frac{2}{\hbar} \int \sum_{\sigma, \mu = \pm} \exp \left[ - \frac{1}{2} \mu \delta t \right]$$

(62)

Hence the force correction is

$$\delta E^{(1)} = -\frac{2}{\hbar} \int B_t \sin \left( \frac{\hbar}{2} \delta t \right) \sin(v t)$$

so that, by using Eq. (29) and performing the calculation of the integrals with arbitrary cutoffs $T_t$ and $\omega_c^{-1} = m/\eta$, one obtains

$$\frac{1}{\eta^R} = \frac{1}{\eta} - \frac{2}{\pi \eta} \left[ \sin \left( \frac{\hbar}{2 \eta} \right) \ln(v/\omega_c) + C + O(v) \right]$$

(64)

where the constant $C$ depends on $T_t$ and $\omega_t$. Although we will not need it below, its detailed form is given in Appendix C in the limit $T_t = 0$.

Consider next second order in $S_{\text{int}}$:

$$i\delta E^{(2)} = \frac{1}{2} \frac{\delta}{\delta t} \left[ S_{\text{int}} \right] = \frac{1}{2} \left[ \frac{\hbar}{2} \right]^2 \sum_{\epsilon, \sigma, \sigma' = \pm} \epsilon_2 \epsilon_4 \int_{\epsilon_2 \to \epsilon_4} B_{i_1, t_1} B_{i_2, t_2} e^{i \sigma_1 (v_{t_1} - v_{t_2}) + i \sigma' (v_{t_2} - v_{t_1})}$$

$$\times \left\{ \exp \left[ - \frac{1}{2} \hbar \delta t \right] \right\}_{0}$$

(65)

Note that $\delta / \delta t$ can be applied also at either $t_1, t_2, t_3$ and all these terms are identical since $\sum_{\epsilon} \delta t$ appears in the same form for all $t_1$, hence a factor of four. Now change all $\epsilon$, $\sigma$, $\sigma'$ to $\epsilon$, $\sigma$, $\sigma'$ and define $\sigma' = \sigma \mu$ to obtain

$$i\delta E^{(2)} = \frac{1}{2} \left[ \frac{\hbar}{2} \right]^2 \sum_{\epsilon, \sigma, \mu = \pm} \epsilon_2 \epsilon_4 \int_{\epsilon_2 \to \epsilon_4} B_{i_1, t_1} B_{i_2, t_2} \sin(v_{t_1 - t_2}) \mu v_{t_3 - t_4}$$

$$\times \exp \left\{ \frac{1}{2} \hbar \delta t \right\}_{0}$$

(66)

where

$$A_2 = \exp \left\{ \frac{1}{2} \hbar \epsilon \left[ - R_{t_1, t_1} + \mu R_{t_1, t_2} - \mu R_{t_2, t_3} \right] \right\} \exp \left\{ \frac{1}{2} \hbar \epsilon \left[ R_{t_2, t_1} + \mu R_{t_2, t_3} - \mu R_{t_3, t_4} \right] \right\}$$

$$\times \exp \left\{ \frac{1}{2} \hbar \epsilon \left[ R_{t_3, t_1} - R_{t_3, t_2} + \mu R_{t_3, t_4} \right] \right\} \exp \left\{ \frac{1}{2} \hbar \epsilon \left[ R_{t_4, t_1} - R_{t_4, t_2} + \mu R_{t_4, t_3} \right] \right\}$$

Note that in $A_2$ if $t_2$ is the maximal time then its second factor equals 1 and $\sum_{\epsilon} \epsilon = 0$. Similarly, if $t_3$ (or $t_4$) is the maximal time, the third (or fourth) factor equals 1 and $\sum_{\epsilon} \epsilon = 0$ (or $\sum_{\epsilon} \epsilon = 0$). Therefore, $t_1$ must be the maximal time and the first factor equals 1. The result is symmetric in $t_3 \leftrightarrow t_4$, so choose $t_3 > t_4$, with factor two. Hence three time orderings, denoted by $A,$
\[ \delta E^{(2)} = \delta E_A + \delta E_B + \delta E_C, \]
\[ \delta E_A = \frac{4}{\hbar^2} \sum_{\mu} \int_{t_3 > t_2 > t_1} \sin \left( \frac{1}{2} h R_{t_1, t_2} \right) \sin \left[ \frac{1}{2} h \left( R_{t_1, t_3} - R_{t_2, t_3} \right) \right] \sin \left[ \frac{1}{2} h \left( R_{t_1, t_4} - R_{t_2, t_4} + \mu R_{t_1, t_4} \right) \right] \times B_{t_1, t_2} B_{t_3, t_4} \sin [v(t_1 - t_2) + \mu v(t_3 - t_4)], \]
\[ \delta E_B = \frac{4}{\hbar^2} \sum_{\mu} \int_{t_3 > t_2 > t_1} \sin \left( \frac{1}{2} h R_{t_1, t_2} \right) \sin \left[ \frac{1}{2} h \left( R_{t_1, t_3} + \mu R_{t_1, t_3} \right) \right] \sin \left[ \frac{1}{2} h \left( R_{t_1, t_4} - R_{t_2, t_4} + \mu R_{t_1, t_4} \right) \right] \times B_{t_1, t_2} B_{t_3, t_4} \sin [v(t_1 - t_2) + \mu v(t_3 - t_4)], \]
\[ \delta E_C = \frac{4}{\hbar^2} \sum_{\mu} \int_{t_3 > t_2 > t_1} \sin \left( \frac{1}{2} h R_{t_1, t_2} \right) \sin \left[ \frac{1}{2} h \left( R_{t_1, t_3} + \mu R_{t_1, t_3} - \mu R_{t_1, t_4} \right) \right] \sin \left[ \frac{1}{2} h \left( R_{t_1, t_4} + \mu R_{t_1, t_4} \right) \right] \times B_{t_1, t_2} B_{t_3, t_4} \sin [v(t_1 - t_2) + \mu v(t_3 - t_4)]. \]

The \( B \) and \( C \) terms can be time ordered as \( A \) by \( t_2 \leftrightarrow t_3 \) in \( B \) and \( t_2 \rightarrow t_1, t_4 \) in \( C \). In terms of the \( \mu = \pm \) components,

\[ \delta E_A^+ + \delta E_C^- = \frac{4}{\hbar^2} \int_A \sin \left( \frac{1}{2} h R_{t_1, t_2} \right) \sin \left[ \frac{1}{2} h \left( R_{t_1, t_3} - R_{t_2, t_3} \right) \right] \sin \left[ \frac{1}{2} h \left( R_{t_1, t_4} - R_{t_2, t_4} + R_{t_1, t_4} \right) \right] \times [B_{t_1, t_2} B_{t_3, t_4} + B_{t_3, t_4}] \sin [v(t_1 - t_2) + v(t_3 - t_4) + t_4] , \]
\[ \delta E_A^- + \delta E_B^+ = \frac{4}{\hbar^2} \int_A \sin \left( \frac{1}{2} h R_{t_1, t_2} \right) \sin \left[ \frac{1}{2} h \left( R_{t_1, t_3} + R_{t_2, t_3} \right) \right] \sin \left[ \frac{1}{2} h \left( R_{t_1, t_4} - R_{t_2, t_4} - R_{t_1, t_4} \right) \right] \times [B_{t_1, t_2} B_{t_3, t_4} + B_{t_3, t_4}] \sin [v(t_1 - t_2) + v(t_4 - t_1) + t_4] , \]
\[ \delta E_B^- + \delta E_C^+ = \frac{4}{\hbar^2} \int_A \sin \left( \frac{1}{2} h R_{t_1, t_2} \right) \sin \left[ \frac{1}{2} h \left( R_{t_1, t_3} + R_{t_2, t_3} \right) \right] \sin \left[ \frac{1}{2} h \left( R_{t_1, t_4} - R_{t_2, t_4} + R_{t_1, t_4} \right) \right] \times [B_{t_1, t_2} B_{t_3, t_4} + B_{t_3, t_4}] \sin [v(t_1 - t_3) + v(t_2 - t_4)]. \]

In Appendix E we derive the \( \ln^2 v \) coefficient directly for the single-cutoff case where \( \tau_0 = 0 \). Here we proceed with a shorter indirect method. In general we have two cutoffs \( m/\eta, \tau_0 \) in Eq. (7) and we define \( \tau_1(m/\eta, \tau_0) \) as the cutoff time for the response \( R_t \) [Eq. (7)]. For the purpose of identifying the leading \( \ln^2 v \) term we take a formal limit such that this cutoff time is \( \tau_1 = 0 \). We will eventually restore physical cutoffs corresponding to \( m/\eta, \tau_0 \) in \( R_t \). The only cutoff for now is \( \tau_0 \) in \( B_{t_1} \) [Eq. (6)]. In this limit \( R_t \rightarrow \frac{1}{\eta} \delta(t)e^{-\delta t} \) where \( \delta \rightarrow +0 \) to ensure the retarded nature [poles of \( 1/(\omega + i\delta) \)].

The significant virtue of this is that the first two equations of (69) vanish since \( R_{t_1, t_4} - R_{t_2, t_4} \rightarrow 0 \), leaving just the last form. The evaluation of \( \delta E^{(2)} \) in this limit is straightforward (Appendix D), leading to

\[ \delta E^{(2)} = \frac{4\eta^2}{\pi^2 h^2} \sin^2 \left( \frac{h}{2\eta} \right) \sin \left( \frac{h}{\eta} \right) v \ln(v\tau_0)[\ln(v\tau_0) + 1]. \]

Hence, from Eq. (29),

\[ \frac{1}{\eta^{(2)}} = \frac{4}{\pi^2 h^2} \sin^2 \left( \frac{h}{2\eta} \right) \sin \left( \frac{h}{\eta} \right) [\ln^2(v\tau_0) + 3 \ln(v\tau_0) + 1]. \]

So far \( \delta E^{(2)} \) is calculated in a formal limit \( \tau_1 \rightarrow 0 \). We proceed by asserting that for any \( \tau_0, \tau_1 \) the leading singularity as \( v \rightarrow 0 \) is a \( \ln^2 v \) term, as expected for a two-loop calculation. This term must involve an \( \eta \)-dependent function \( f_\eta(\tau_0, \tau_1) \) that has dimensions of time. Fixing the coefficient of \( \ln^2[vf_\eta(\tau_0, \tau_1)] \) as in Eq. (71), we have \( f_\eta(v\tau_0, \tau_1) = \tau_0 \) while for \( \tau_0 \rightarrow 0 \), when \( \tau_1 \rightarrow m/\eta = 1/\omega_c \), we must have the form \( f_\eta(0, \tau_1) = b(\eta)(\tau_1) = b(\eta)/\omega_c \). The two-loop correction (71) becomes, at \( \tau_0 = 0 \),

\[ \frac{1}{\eta^{(2)}} = \frac{4}{\pi^2 h^2} \sin^2 \left( \frac{h}{2\eta} \right) \sin \left( \frac{h}{\eta} \right) \ln^2 \left( \frac{v}{\omega_c} \right) + O(\ln v). \]

The renormalized friction therefore has the form

\[ \frac{1}{\eta^R} = \frac{1}{\eta} - \frac{2}{\pi^2} \sin \left( \frac{h}{2\eta} \right) \ln \left( \frac{v}{\omega_c} \right) + \frac{4}{\pi^2 h^2} \sin^2 \left( \frac{h}{2\eta} \right) \sin \left( \frac{h}{\eta} \right) \times \left\{ \ln^2 \left( \frac{v}{\omega_c} \right) + b_0(\eta) \ln \left( \frac{v}{\omega_c} \right) \right\}. \]

We have thus identified the coefficient of the \( \ln^2(v) \) term; this coefficient is also identified by the more lengthy calculation of the \( \tau_0 = 0 \) case in Appendix E. In Appendix E we furthermore show that the coefficient of the \( \ln v \) term [i.e., \( \sin^2 \left( h/(2\eta) \right) \sin \eta/b_0(\eta) \)], has at least one factor of \( \sin(h/(2\eta)) \). Hence the perturbation expansion as well as the following RG analysis are justified near the zeros of \( \sin(h/(2\eta)) \).

We note that in the semiclassical limit the perturbation expansion is in \( R^{2n-1} B^n/\eta^2 \sim 1/\eta^{n+1} \) for large \( \eta \); in the quantum case the \( R^{2n-1} \) factors become periodic functions. The main conclusion is that there is a new small parameter in the perturbation series, \( \sin(h/(2\eta)) \).
B. Perturbations from $S_c$

Here we consider the $S_c$ interaction in Eq. (21). The $S_c$ terms are

$$\langle \delta_t \theta, S_c \rangle = \langle \delta_t \theta, S_c^2 \rangle = 0. \quad (74)$$

However, the mixed term and the corresponding correction to $1/\eta$ are

$$\delta R^m_{\ell,r} = i \langle \delta_t \theta, S_c S_m \rangle,$$

$$\Rightarrow \frac{1}{\eta^m} = \frac{2}{\pi \hbar} \left[ \frac{\sin \frac{\hbar}{2\eta} \left( \sin \frac{\hbar}{\eta} - \frac{\hbar}{\eta} \right)}{\hbar} \right] \sin (\nu \tau_1),$$

which does not vanish at sin $(\hbar/(2\eta)) = 0$. Note, however that this term is $\sim \hbar^3$ (i.e., a three-loop term). Furthermore, other response functions do show such zeros. For example, for the $R_{t,t}$ correlation [Eq. (77) below] we have $\langle \delta_t \sin \frac{\hbar}{2\eta} S_c \rangle = 0$ to first order, while in second order,

$$\delta R^m_{t,t} = \frac{2i}{\eta^2} \langle \delta_t \sin \frac{\hbar}{2\eta} S_m \rangle,$$

$$\Rightarrow \frac{1}{\eta^2} = \frac{2}{\pi \hbar} \left[ \frac{\sin \frac{\hbar}{\eta} \left( \sin \frac{\hbar}{\eta} - \frac{\hbar}{\eta} \right)}{\hbar} \right] \sin (\nu \tau_1). \quad (76)$$

C. Renormalization-group analysis

We note that there are many operators that have vanishing perturbative terms at sin $(\hbar/(2\eta)) = 0$ to second order in $S_m$, $S_c$; for example, the dissipation term in Eq. (9) $\langle \delta_t \sin (\hbar/2\eta) \rangle$, or the response to an ac field with frequency $\nu$ $\langle \delta_t \cos \theta_t \sin \frac{\hbar}{2\eta} \rangle$.

We propose then that a RG consistent theory corresponds to

$$\frac{h}{2\eta^R} = \frac{h}{2\eta} - \frac{2}{\pi} \sin^2 \left( \frac{h}{2\eta} \right) \ln[\tau_1 \nu] + \frac{4}{\pi^2} \sin^3 \left( \frac{h}{2\eta} \right) \times \cos \left( \frac{h}{2\eta^R} \right) \left[ \ln^2[\tau_1 \nu] + b_0(\eta) \ln[\tau_1 \nu] \right]. \quad (79)$$

Taking a sine of both sides it yields to order $g^3$, with $b_0 = b_0(\eta) = 0$,

$$g_R = g + (g^n \nu / \omega_c) + g^3 \ln^2(\nu / \omega_c) + b_0(\eta) \ln(\nu / \omega_c), \quad (80)$$

where $\pm$ refers to $g = 0$ with $\cos (h/(2\eta)) = \pm 1$, leading to

$$\beta(g) = \frac{dg_R}{dv} = \pm g^2 - b_0 g^3 + O(g^4). \quad (81)$$

This RG equation is satisfied for both $\pm$ fixed points as seen by substituting Eq. (80). We propose then that $g = 0$ are exact zeros of the perturbation expansion and the additional requirement of a RG structure leads to the result (80).

Equation (80) yields fixed points at $\hbar/(2\eta_n) = n\pi$ with $n = 1, 2, 3, \ldots$ that are attractive at $\eta > \eta_n$ and repulsive at $\eta < \eta_n$ (i.e., the flow of $\eta$ is always smaller than $\eta$). At these fixed points a Gaussian evaluation yields the correlation $\langle \cos \theta_t \cos \theta_0 \rangle \sim t^{-2n}$. We recall now a theorem for the lattice model31 where the equilibrium action with mass-related cutoff is replaced by an action on a lattice resulting in an $XY$ model with long-range interactions. The theorem states31 that $\langle \cos \theta_t \cos \theta_0 \rangle \sim 1/t^{2n}$; this result was also derived9 to first order in $\eta$. The range $\eta > \eta_1$ has a RG flow to $\eta_1$ and is therefore consistent with the theorem. The hypothesis of Gaussian fixed points corresponding to $n \geq 2$ is inconsistent with the theorem; that is, $\langle \cos \theta_t \cos \theta_0 \rangle$ becomes a relevant operator at the $n = 2$ points rendering them unstable. Note that in the SEB problem the fixed point $\theta_t$ corresponds to a lead-dot voltage and its correlations determine the SET conductance,11,12,21 while in the ring problem it corresponds to fluctuations in the circular asymmetry.

For $\eta < \eta_1$ the system could have non-Gaussian fixed points or a line of fixed points as hinted by the small $\eta$ perturbation.9 The equilibrium $K_1(\phi_t)$ was evaluated for small $\eta$ and for $T \to 0$ has the form9 $K_1(\phi_t) \sim \delta(\phi_t - 1/2) / T$ (i.e., the dissipation is concentrated at the single point $\phi_t = 1/2$). This implies from Eq. (39) that $\eta^R \sim T$ and therefore vanishes at temperature $T = 0$. It is not clear, however, that $\eta = 0$ is a fixed point in the RG sense and if so what is its range of attraction. An $\eta = 0$ fixed point would imply the implausible result that the ring conductance diverges for small but finite $\eta$. We therefore expect that $\eta_1 \equiv \eta_1^R$ is the single fixed point in this system, as illustrated in Fig. 4.

V. DISCUSSION

The special value $g^R = \hbar/(2\pi)$ has a topological interpretation as a Thouless charge pump,26 as shown in the introduction.

![FIG. 4. (Color online) RG flow of $\eta$.](image_url)
Hence a slow change in $\phi_s$ by one unit results in transporting a unit charge once around the ring if $\eta_R = h/(2\pi)$. Such quantization has been shown for cases where the spectrum has a gap,\textsuperscript{26} although quantized charge transport was shown also in cases without a gap.\textsuperscript{32,33} In our case the gap vanishes\textsuperscript{15} at flux $\phi = 1/2$. Vanishing of this gap is essential in solving the dissipation problem in the ring via Landau-Zener transitions, as studied in related models.\textsuperscript{34} We note that the quantized $\eta_R$ also results from arguing that there should be a unique frequency $\omega_K = (2\pi/h)E = v$ as $E \to 0$ [see discussion below Eq. (38)], as suggested by linear response.

We conclude from Eq. (45) that for $\eta > \eta_1 \equiv \eta_R$ the SSB satisfies the quantization (see definitions in Sec. II E)

$$\int_0^1 \frac{C^2_0(N_0)}{C_k^2} R_q(N_0) dN_0 = \frac{h}{e^2}. \tag{82}$$

In particular, when $\eta/h > 1$ we have\textsuperscript{6-9} from the known $M^*/m \sim e^2h/\eta$ and from Eq. (6) $C_0/C_k = 1 + O(e^{-2\eta/h})$. We expect $R_q$ to be independent of $N_0$ at large $\eta$, hence

$$R_q = \frac{h}{e^2}[1 + O(e^{-\eta/h})], \tag{83}$$

similar to the $N_c = 1$ case.\textsuperscript{3}

The conductance of the ring can be defined by the voltage around the ring $2\pi E/e$ and the current $e(\vartheta )/(2\pi)$, hence we predict that the conductance for $\eta > \eta_R$ is

$$G_{\text{ring}} = \frac{e^2}{4\pi^2\eta_R} = \frac{e^2}{h}. \tag{84}$$

While this well-known quantum conductance seems natural, we emphasize that it is due to the inherent nonequilibrium nature of the driving force and the specific limiting procedure of taking a dc limit before the linear response limit [Eq. (39)].

Finally, we consider the conditions for our proposed box experiment. The Coulomb box (i.e., a metallic quantum dot) should be connected to the electrode with $N_c \gg 1$ degenerate channels; in fact $N_c$ can be fairly small and yet reproduce the $N_c \to \infty$ case, except at exponentially small temperatures.\textsuperscript{35} By analogy with $E = h\phi_s$ in the ring, we propose measuring the response to a gate voltage that is linear in time $N_0 = Et$. This leads to a dc current into the Coulomb box whose dissipation is the average in Eq. (45). The field $E$ should be sufficiently small so that $g_R$ is sufficiently near the fixed point. For an initial $g \approx 1$ integration of $\delta g_R/\delta \ln E = g^\prime_R$ yields $g_R = 1/(\ln(h\omega_c/E) \ll g$. For example, for $g_R \lesssim 0.1$ and a typical $h\omega_c \approx 1$ meV one needs $E/h \lesssim 10^8$ Hz. $E/h$ has frequency units, corresponding to $10^8$ electrons/s flowing into the box.

While it may be possible to measure dissipation directly (e.g., via heating), we propose measuring instead the charge fluctuations (noise) $S_Q(\omega) = e^2(N_c N_t)_\omega$. The latter should be measured at frequency, temperature, and level spacings $\Delta$ such that $\Delta < \omega, T \ll 10^8$ Hz, to yield the response to the force $E$. FDT relates the (symmetrized) noise and the retarded response $K(\omega)$ [Eq. (41)] via $S_Q(\omega) = h\coth(h\omega/(2T))\text{Im}K(\omega)$. From Eq. (45) we have (at $T = 0$) that the gate voltage averaged noise $\tilde{S}_Q(\omega)$ satisfies $\tilde{S}_Q(\omega)(\frac{2e^2}{\omega})^2/\omega = h/\eta_R$. In particular, as the fixed point is approached we predict $\tilde{S}_Q(\omega)(\frac{2e^2}{\omega})^2/\omega = 2\pi$.

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\section*{APPENDIX A: MAPPING COULOMB BOX AND RING}

The AES mapping has been extensively used, yet we find it useful to reproduce it since the relation between correlation functions has received less attention.

The Coulomb-box action corresponding to the Hamiltonian (40) is

\begin{equation}
-i\hbar S = \int \left\{ \sum_{\alpha} d_{\alpha,i}^\dagger (i\hbar \partial_t - \epsilon_\alpha) d_{\alpha,i} - E_c(\hat{N} - N_0)^2 \right\} - i\hbar S_{\text{lead}} - i\hbar S_{\text{tunn}},
\end{equation}

\begin{equation}
-i\hbar S_{\text{lead}} = \int \sum_k d_{k,i}^\dagger (i\hbar \partial_t - \epsilon_k) d_{k,i}, \quad -i\hbar S_{\text{tunn}} = \int \sum_{k,\alpha} l_{k,\alpha,i} d_{k,i}^\dagger + \text{H.c.,}
\end{equation}

with the partition $Z = e^{-S}$. Adding a variable $\vartheta_t$ to the path integral yields

\begin{equation}
-i\hbar S = \int \left\{ E_c \left[ \hat{N} - N_0 - \frac{\hbar}{2E_c} \vartheta_t \right]^2 + \sum_{\alpha} d_{\alpha,i}^\dagger (i\hbar \partial_t - \epsilon_\alpha) d_{\alpha,i} - E_c(\hat{N} - N_0)^2 \right\} - i\hbar S_{\text{lead}} - i\hbar S_{\text{tunn}}
= \int \left\{ \sum_{\alpha} d_{\alpha,i}^\dagger (i\hbar \partial_t - \epsilon_\alpha - \hbar \vartheta) d_{\alpha,i} + \frac{1}{4E_c} \left[ i\hbar \vartheta_t + 2E_c N_0 \right]^2 \right\} - i\hbar S_{\text{lead}} - i\hbar S_{\text{tunn}}. \tag{A2}
\end{equation}
Now define $d_a = e^{-i\theta} \hat{d}_a$:

$$-i\hbar S = \int \left\{ \sum_a \hat{d}^\dagger_a (i\hbar \partial_\tau - \epsilon_a)\hat{d}_a + \frac{\hbar^2}{4E_c} \dot{\theta}^2 + \dot{\theta} T N_0 + \sum_{k,a,i} \{ t_{k,a,i} \hat{d}^\dagger_a t_{k,a,i} e^{i\theta} + \text{H.c.} \} \right\} - i\hbar S_{\text{lead}}. \tag{A3}$$

The ring action in terms of $\theta_t$ is derived by integrating out the fermions $\hat{d}_a$ and $a_k$. Define time-ordered Greens’ functions on the dot $G_{0k,i}(\omega) = (\omega - \epsilon_{k,i} + i\text{sgn} \omega \delta_{k,0})^{-1}$ and on the lead $G_{0k,i}(\omega) = (\omega - \epsilon_{k,i} + i\text{sgn} \omega \delta_{k,0})^{-1}$. In matrix notation,

$$\hat{G}_i^{-1}(t,t') = \begin{pmatrix} G_{0a,i}(t,t') & 0 \\ 0 & G_{0k,i}(t,t') \end{pmatrix} + \begin{pmatrix} 0 & t_{k,a,i} e^{i\theta} \\ t^*_{k,a,i} e^{-i\theta} & 0 \end{pmatrix} \delta(t - t') \equiv \hat{G}_{0i}^{-1} + \hat{T}_i. \tag{A4}$$

The trace over fermions, using det($i\hat{G}$) = exp(Tr ln $i\hat{G}$), yields

$$S_{\text{eff}} = -\sum_i \text{Tr} \ln i\hat{G}_i^{-1}(t,t) = -\sum_i \text{Tr} \ln \{ i\hat{G}_i^{-1}(t,t')[\delta(t - t') + \hat{G}_0(t',t)\hat{T}_i(t)] \}. \tag{A5}$$

Expanding in $\hat{T}$, the zeroth order is $\theta_t$ independent, the first order vanishes, hence to second order,

$$S_{\text{eff}} = -\frac{1}{2} \sum_i \text{Tr} \{ \hat{G}_0 \hat{T} \hat{G}_0 \hat{T} \} = -\frac{1}{2} \sum_i \int_{t'} \{ G_{0a,i}(t,t') G_{0k,i}(t',t) [t_{k,a,i}]^2 e^{i\theta - i\theta_t} + \text{H.c.} \}. \tag{A6}$$

For completeness we derive the Matsubara effective action using $\sum_a G_{a,i}(\tau) = T \sum_n G(\omega_n) e^{i\omega_n \tau}$ with fermionic $\omega_n = \pi T (2n + 1)$:

$$G(\omega_n) = \int_0^\infty \rho_{\text{dot}}(\epsilon) \left[ \frac{1}{i\omega_n - \epsilon} - \frac{1}{i\omega_n + \epsilon} \right] = \int_0^\infty \rho_{\text{dot}}(\epsilon) \frac{2i\omega_n}{\omega_n^2 + \epsilon^2} = -i\pi \rho_{\text{dot}}(0) \text{sgn}(\omega_n), \tag{A7}$$

$$\sum_a G_{0a,i}(\tau) = 2\pi \rho_{\text{dot}}(0) \sum_{n=0}^\infty \sin(\omega_n \tau) = \rho_{\text{lead}}(0) \frac{\pi T}{\sin(\pi T \tau)},$$

where $\rho_{\text{dot}}(\epsilon)$ is the dot density of states, assumed symmetric, and eventually constant. With the lead density of states $\rho_{\text{lead}}(\epsilon)$, and assuming a constant $t_{k,a,i}$,

$$S_{\text{eff}} = -\frac{1}{2} [\tilde{\tau}]^2 N_c \rho_{\text{lead}}(0) \rho_{\text{lead}}(0) \int \int \frac{\pi^2 T^2}{\sin^2(\pi T (\tau - \tau'))} \cos[\theta(\tau) - \theta(\tau')] \tag{A8},$$

where $N_c = \sum_j$ is the number of channels. This is the well-known equilibrium ring system with a bosonic CL environment, where $\eta = \frac{1}{2}(\frac{1}{2} N_c \rho_{\text{dot}}(0) \rho_{\text{lead}}(0))$ and $m = 1/(2E_c)$. The expansion in $\hat{T}$ is justified for $|t| \rightarrow 0$; however, with $N_c \rightarrow \infty$ any value of $\eta$ can be generated. In fact $N_c$ can be fairly small and yet reproduce the $N_c \rightarrow \infty$ case, except at exponentially small temperatures. A similar derivation holds for the Keldysh action leading to the form (21).

We proceed now to map observables of the Coulomb box to those of the ring problem. Since the action (A3) has a term $+\dot{\theta} N_0$ we identify $N_0 = -\phi_0$ where $\phi_0$ is the flux through the ring (in units of the quantum flux). Hence,

$$\hbar \theta(\dot{\theta}) = \int \theta(\dot{\theta}) \exp \left[ -\frac{i}{\hbar} \int E_c \left( \dot{N} - N_0 - \frac{\hbar^2}{2E_c} \right)^2 + \text{fermion terms} \right] = \int \theta(\dot{\theta} + 2E_c \dot{N} - 2E_c N_0) \exp \left[ -\frac{i}{\hbar} \int \frac{\hbar^2}{4E_c} \dot{\theta}^2 + \text{fermion terms} \right] = 2E_c (\langle \tilde{N} \rangle - N_0). \tag{A9}$$

In particular, without interaction, $t_{k,a} = 0$, the charge has no fluctuations $\langle \tilde{N} \rangle = 0$ (for $|N_0| < \frac{1}{2}$) so that $\hbar \theta(\dot{\theta}) = -2E_c N_0 = 2E_c \phi_0$. Consider next the time-ordered $T$ correlations (the following is the same for $(\dot{\theta}_t^+ \dot{\theta}_t^+)$, $(\dot{\theta}_t^- \dot{\theta}_t^-)$ with $\pm$ Keldysh contours),

$$\hbar^2 T(\dot{\theta}_t \dot{\theta}_t) = \int \theta(\dot{\theta}_t \dot{\theta}_t) \exp \left[ -\frac{i}{\hbar} \int E_c \left( \dot{N} - N_0 - \frac{\hbar^2}{2E_c} \dot{\theta}^2 + \text{fermion terms} \right] = \int \theta(\dot{\theta}_t + 2E_c \dot{N}_t - 2E_c N_0)(\dot{\theta}_t + 2E_c \dot{N}_t - 2E_c N_0) \exp \left[ -\frac{i}{\hbar} \int \frac{\hbar^2}{4E_c} \dot{\theta}^2 + \text{fermion terms} \right] = \hbar^2 T(\dot{\theta}_t \dot{\theta}_t) + 4E_c^2 T(\langle \tilde{N}_t - N_0 \rangle (\tilde{N}_t - N_0)). \tag{A10}$$
To obtain the retarded response, 

$$-iD_{t,t'}^{R} = \theta(t - t')\langle [A_t, B_{t'}] \rangle = \theta(t - t')\langle A_{t'} B_{t'} - B_{t'} A_t \rangle = T(A_{t'}^+ B_{t'}^+) - (B_{t'}^+ A_t^+), \tag{A11}$$

where ± are Keldysh contour indices, so that $A^+$ is earlier than $B^-$.

Define the response $K_{t,t'}$ of the Coulomb box, as well as the response of ring problem $\tilde{K}_{t,t'}$ in the form (displayed here with operators whose $A_t = 0$ to allow relation with time ordering)

$$\tilde{K}_{t,t'} = +i\theta(t - t')\langle [\hat{\theta}_t - \langle \hat{\theta} \rangle, \hat{\theta}_{t'} - \langle \hat{\theta} \rangle] \rangle, \quad K_{t,t'} = +i\theta(t - t')\langle [\hat{N}_t - \langle \hat{N} \rangle, \hat{N}_{t'} - \langle \hat{N} \rangle] \rangle. \tag{A12}$$

From Eq. (A10) we have

$$\hbar^2 K_{t,t'} = -2E_r \hbar \delta(t - t') + 4E_z^2 K_{t,t'}, \tag{A14}$$

which is reproduced as Eq. (40). This relation is consistent with results in Ref. 22.

**APPENDIX B: SEMICLASSICAL CASE: FIRST AND SECOND ORDER**

### 1. First-order term

First-order perturbation of the Green's function

$$R_{t,t'}^{(1)} = -\frac{1}{2} \int_{t_1,t_2} B_{t_1,t_2} \Delta_{\sigma_1,\sigma_2,\alpha_1,\alpha_2} \exp \left( i\alpha_1 (-\sigma R_{t_1,t_2} + \alpha_4 R_{t_1,t_2}) + i\alpha_2 (\sigma R_{t_1,t_2} - \alpha_4 R_{t_1,t_2}) + i\alpha_3 (\sigma R_{t_1,t_2} - \sigma R_{t_1,t_2} + \alpha_4 R_{t_1,t_2}) \right) e^{i\sigma \psi(t_1-t_2)} \tag{B1}$$

An averaging with Gaussian weight

$$\langle e^{i\beta_1 + i\beta_2 + \ldots + i\beta_4 + i\beta_4 + \ldots} \rangle = e^{i\beta_1 + i\beta_2 + \ldots + i\beta_4 + i\beta_4 + \ldots} = e^{i\beta_1 + i\beta_2 + \ldots + i\beta_4 + i\beta_4 + \ldots} \tag{B2}$$

The retarded function

$$R_{t,t'}^{(1)} = \frac{1}{4\pi} \int_{t_1,t_2} \sum_{\sigma_1,\sigma_2} \partial_{\sigma_1} B_{t_1,t_2} \exp \left( i\alpha_1 (-\sigma R_{t_1,t_2} + \alpha_4 R_{t_1,t_2}) + i\alpha_2 (\sigma R_{t_1,t_2} - \alpha_4 R_{t_1,t_2}) + i\alpha_3 (\sigma R_{t_1,t_2} - \sigma R_{t_1,t_2} + \alpha_4 R_{t_1,t_2}) \right) e^{i\sigma \psi(t_1-t_2)} \tag{B3}$$

In the last expression we use $R_{t_1,t_2} = 0$.

### 2. Second-order term

Using the same procedure for the second order:

$$R_{t,t'}^{(2)} = \frac{i}{2} \langle \hat{\theta}_t, \hat{\theta}_s \rangle (S_{t_1})^2 = -\frac{i}{8} \int_{t_1,t_2,t_3,t_4} B_{t_1,t_2} B_{t_3,t_4} \langle \hat{\theta}_t \hat{\theta}_s \hat{\theta}_{t_1} \hat{\theta}_{t_2} \rangle \cos \left( \theta_{t_1} - \theta_{t_2} \right) \hat{\theta}_{t_3} \hat{\theta}_{t_4} \cos \left( \theta_{t_3} - \theta_{t_4} \right) \hat{\theta}_{t_1} \hat{\theta}_{t_2} \tag{B4}$$

using the symmetry between $\sigma_1 \leftrightarrow -\sigma_1$ and $t_1 \leftrightarrow t_2$ similarly for $t_3, t_4$:

$$R_{t,t'}^{(2)} = \frac{1}{8} \int_{t_1,t_2,t_3,t_4} B_{t_1,t_2} B_{t_3,t_4} e^{i\psi(t_2-t_1) - i\psi(t_3-t_4)} \partial_{\alpha_1} \left[ -R_{t_1,t_2} + R_{t_1,t_2} - R_{t_1,t_2} + \alpha_6 R_{t_1,t_2} \right] \left[ R_{t_3,t_4} - R_{t_3,t_4} + \alpha_6 R_{t_3,t_4} \right] \times \left[ R_{t_2,t_3} - R_{t_2,t_3} - R_{t_2,t_3} + \alpha_6 R_{t_2,t_3} \right] \left[ R_{t_4,t_5} - R_{t_4,t_5} + \alpha_6 R_{t_4,t_5} \right] \tag{B5}$$
the choice \( t_1 > t_2, t_3, t_4 \), only \( R_{t_3,t_4} \) remains. \( R_{t} \) is real, we separate the exponent into two sine and two cosine terms as follows:

\[
R_{t,t'}^{(2)} = \frac{1}{8} \int_{t_1,t_2,t_3,t_4} B_{t_1,t_2} B_{t_3,t_4} \{ \cos v(t_1 - t_2) \cos v(t_3 - t_4) - \sin v(t_1 - t_2) \sin v(t_3 - t_4) \} R_{t_1,t_2} \left[ R_{t_3,t_4} + R_{t_3,t_2} - R_{t_2,t_4} \right] \\
\times \left[ R_{t_1,t_3} - R_{t_2,t_3} - R_{t_1,t_4} + R_{t_2,t_4} \right] \left[ R_{t_1,t'} - R_{t_2,t'} + R_{t_3,t'} - R_{t_4,t'} \right].
\] (B6)

This long multiplicity of \( R_{t} \) terms is now separated into eight different terms. For the terms with the cosine we calculate explicitly three terms, which we label by \( a \) to \( c \). Term “a” is

\[
R_{t,t'}^{a} = \frac{1}{2} \int_{t_1,t_2,t_3,t_4} B_{t_1,t_2} \cos v(t_1 - t_2) R_{t_1,t_2} R_{t_3,t_4} \left( R_{t_1,t'} - R_{t_2,t'} \right) B_{t_3,t_4} \cos v(t_3 - t_4) \left( R_{t_1,t_3} - R_{t_2,t_3} \right) \left( R_{t_1,t_4} - R_{t_2,t_4} \right)
\] (B7)

This term in \( \omega \) space

\[
R_{\omega}^{a} = -\frac{1}{2} R_{\omega}^{2} \int_{t} R_{t} B_{t} \cos v (\epsilon^{ \text{tot} } - 1) \tilde{C}_{t},
\]

with \( \tilde{C}_{t} = 2 (C_{t=0}^{(1)} - C_{t}^{(1)}) \). Similarly we choose two different terms “b” and “c” and write them directly in \( \omega \) space:

\[
R_{\omega}^{b} = R_{\omega}^{2} \int_{t} R_{t}^{(11)} B_{t} \cos v (\epsilon^{ \text{tot} } - 1),
\]

\[
R_{\omega}^{c} = R_{\omega}^{2} \left[ \int_{t} R_{t} B_{t} \cos v (\epsilon^{ \text{tot} } - 1) \right]^{2} = R_{\omega}^{2} \left( R_{\omega}^{(11)} \right)^{2}.
\]

Note the \( R_{\omega}^{(11)} \) in the expression \( R_{\omega}^{\text{R}} \) is the first-order result of the retarded Green function. \( R_{\omega}^{\text{R}} \) is the reducible term containing multiplication of \( R_{t}^{(11)} \). Renormalized \( \eta \) for small \( v \) is

\[
\frac{1}{\eta_{2}^{2}} = \frac{1}{2} \eta_{2}^{2} \int_{t} R_{t} B_{t} \tilde{C}_{t}(t) = \frac{\hbar}{\pi \eta_{2}^{2}} \int_{t} R_{t} B_{t} \{ \ln t + \gamma + O(v) + O(1/t) \} = \frac{\hbar}{2 \pi \eta_{2}^{2}} \ln v + O(v),
\]

\[
\frac{1}{\eta_{2}^{2}} = -\frac{\hbar}{\pi \eta_{2}^{2}} \int_{t} R_{t}^{(11)} B_{t} = -\frac{\hbar}{2 \pi \eta_{2}^{2}} \int_{t} R_{t} B_{t} \{ \ln t + \gamma + 1 + O(v) + O(1/t) \} = \frac{\hbar}{2 \pi \eta_{2}^{2}} \ln v - \frac{\hbar}{2 \pi \eta_{2}^{2}} \ln v + O(v),
\]

\[
\frac{1}{\eta_{2}^{2}} = \int_{t} R_{t} B_{t} \eta_{2}^{2} = \frac{\hbar^{2}}{2 \pi \eta_{2}^{4}} \{ \ln v \ln O(v) \} = \frac{\hbar^{2}}{2 \pi \eta_{2}^{4}} \ln v + O(v).
\]

The terms containing the sine in Eq. (B6) are, in general, of order \( O(v) \); however, we have identified the following term which, depending on the order of limits, may contribute a term logarithmic in \( v \) for small \( v \):

\[
R_{\omega}^{d} = -R_{\omega}^{2} \int_{t_1,t_2} R_{t_1} R_{t_2} R_{t_1} \sin v t_1 \sin v t_2 (1 - e^{i \omega t_3}) \int_{t_3} \left( R_{t_3 + t_1} - R_{t_3} \right).
\] (B11)

We label the dissipation parameter form this term by \( \delta(1/\eta_{2}^{2}) = \lim_{\omega \to 0} (i \omega) R_{\omega}^{d} \) and find the logarithmic prefactor in Eq. (56), where we use for \( t_1 > 0 \)

\[
\int_{t_1} \left( R_{t_1 + t_1} - R_{t_1} \right) = \frac{1}{\eta} \int_{-t_1}^{0} (1 - e^{-\delta(1/\eta_{2}^{2}) \tau}) + \frac{1}{\eta} \int_{0}^{\infty} (e^{-\delta(1/\eta_{2}^{2}) \tau} - e^{-\delta(1/\eta_{2}^{2}) \tau}) \equiv \frac{\hbar}{\eta}
\] (B12)

APPENDIX C: QUANTUM CASE: FIRST ORDER, MORE DETAILS

Let us give the detailed calculation of the first-order correction in the case of a mass-only cutoff (i.e., \( \tau_0 = 0 \)). Taking the derivative of Eq. (63) in the text we have

\[
\partial_{\tau} \tilde{E}^{(1)} = -\frac{2}{\hbar} \int_{\tau > 0} \tau B(\tau) \sin \left( \frac{\hbar}{2} R(\tau) \right) \cos(\nu \tau) = \frac{2\hbar}{\pi} \int_{\tau > 0} d\tau \frac{\sin \left( \frac{\hbar}{2\eta} (1 - e^{-\frac{\hbar}{\eta} \tau}) \right) \cos(\nu \tau)}{\nu}
\]

\[
= \frac{2\hbar}{\pi} \left[ \sin \left( \frac{\hbar}{2\eta} \right) \int_{\tau > 0} d\tau \left( 1 - e^{-\frac{\hbar}{\eta} \tau} \right) \cos(\nu \tau) - \int_{\tau > 0} d\tau \left[ \sin \left( \frac{\hbar}{2\eta} (1 - e^{-\frac{\hbar}{\eta} \tau}) \right) - \sin \left( \frac{\hbar}{2\eta} \right) (1 - e^{-\frac{\hbar}{\eta} \tau}) \right] \cos(\nu \tau) \right]
\]

\[
= \frac{2\hbar}{\pi} \left[ \sin \left( \frac{\hbar}{2\eta} \right) \ln \left( \frac{\eta}{mv} \right) + f \left( \frac{\hbar}{2\eta} \right) + O(\nu) \right],
\] (C1)
since the first integral can be computed exactly and in the second one we can set $v = 0$ to get the constant piece. This determines the constant $C = f(h/(2\eta))$ given in the text in Eq. (64), where the function $f(x)$ is defined as

$$f(x) = \int_0^{+\infty} \frac{dt}{t} \sin[x(1 - e^{-t})] - \sin(x(1 - e^{-t})) = -\int_0^{1} \frac{dz}{(1 - z) \ln(1 - z)} \sin(xz) - x = \frac{1}{6} x^3 \ln\left(\frac{8}{3}\right) + O(x^5)$$

and is a nicely convergent integral, where one can rescale $t$ freely. Although it is not periodic in $x$, upon plotting it one notes that it seems to become almost periodic at large $x$.

**APPENDIX D: QUANTUM CASE: SECOND ORDER FOR $\tau_1 \to 0$**

Since $\sin(\frac{1}{2}hR_{t_1,t_2})$ is a retarded function, we use for $R_t = \Theta(t)e^{-\delta t}$

$$\sin\left(\frac{1}{2}\hbar R_{t_1,t_2}\right) \rightarrow \sin\left(\frac{\hbar}{2\eta}\right)e^{-\delta(t_1-t_2)}, \quad \sin\left[\frac{1}{2}\hbar(R_{t_1,t_2} + R_{t_2,t_1})\right] \rightarrow \sin\left(\frac{\hbar}{\eta}\right)e^{-\delta(t_1-t_2)-\delta(t_2-t_1)},$$

$$\sin\left[\frac{1}{2}\hbar(R_{t_1,t_2} - R_{t_1,t_4} + R_{t_2,t_4})\right] \rightarrow \sin\left(\frac{\hbar}{2\eta}\right)e^{-\delta(t_1-t_3)-\delta(t_4-t_3)},$$

(D1)

For example, the Fourier transform of $t_1 - t_3$ and $t_2 - t_3$ should have $1/(x + i\delta(x_2 + i\delta))$. Define the variables

$$t_2' = t_2 - t_1, \quad t_3' = t_3 - t_2, \quad t_4' = t_4 - t_3, \quad \Rightarrow t_2 = t_2' + t_1, \quad t_3 = t_3' + t_1, \quad t_4 = t_4' + t_2' + t_1.$$  

(D2)

These variables are more convenient since their range is independent $-\infty < t_2', t_3', t_4' < 0$. The product of all convergence factors is then $e^{\delta(3t_2' + 4t_3' + 3t_4')}$, with the factors 3, 4, and 3 unimportant since $\delta \to 0$. Hence

$$\delta E^{(2)} = \frac{4}{\hbar^3} \sin^2\left(\frac{\hbar}{2\eta}\right) \sin\left(\frac{\hbar}{\eta}\right) \int_{(\omega_1,\omega_2)} B_{\omega_1,\omega_2} \sum_{\sigma = \pm} \frac{\sigma}{2!} \int_{A} e^{i\sigma(x_2(t_1 - t_2'))} e^{i\sigma x(t_3 + t_4')} \delta(t_2' + t_4')\delta(t_1 - t_2')$$

$$\times \frac{1}{(-2\sigma v + i\omega_1 + i\omega_2 + \delta)(-\sigma v + i\omega_2 + \delta)}$$

$$= \frac{4}{\hbar^3} \sin^2\left(\frac{\hbar}{2\eta}\right) \sin\left(\frac{\hbar}{\eta}\right) \sum_{\sigma = \pm} \frac{\sigma}{2!} \int_{\omega_1} \frac{1}{(\omega_1 - \sigma v - i\delta)^2} \frac{1}{1 + \omega_1^2 \tau_0^2} \int_{\omega_2} \frac{1}{(\omega_2 - \sigma v - i\delta)^2} \frac{1}{1 + \omega_2^2 \tau_0^2},$$

(D3)

where the integral over $\omega_2$ gives

$$\int_0^{\infty} \frac{d\omega_2}{\omega_2 - \sigma v - i\delta} \frac{\omega_2}{1 + \omega_2^2 \tau_0^2} = 2\sigma v \int_0^{\infty} \frac{d\omega_2}{\omega_2 - \sigma v - i\delta} = -\sigma v \ln(v\tau_0) + O(v^3 \tau_0^2 \ln(v\tau_0)),$$

(D4)

and over $\omega_1$ gives

$$\int_0^{\infty} \frac{d\omega_1}{(\omega_1 - \sigma v - i\delta)^2} \frac{\omega_1}{1 + \omega_1^2 \tau_0^2} = 2 \int_0^{\infty} d\omega_1 \left[\frac{\omega_1}{(\omega_1 - \sigma v - i\delta)^2} + \frac{2 \omega_1^2}{(\omega_1 - \sigma v - i\delta)^2(\omega_1 + \sigma v + i\delta)^2}\right] = -2 \ln(v\tau_0) - 2,$$

(D5)

where in the last integral $\tau_0 \to 0$ can be taken. Substituting (D4) and (D5) into (D3) leads to the result (70).

**APPENDIX E: QUANTUM CASE: SECOND ORDER WITH MASS CUTOFF**

In this Appendix we rederive the second-order quantum case using directly a mass cutoff. In particular, we identify the coefficient of the $\ln^2 v$ term, confirming that coefficient in Eq. (70), and derive some properties of the second-order $\ln v$ term. We express Eq. (66) as

$$\delta E^{(2)} = \frac{i}{4\hbar^3} \sum_{\epsilon_i} \epsilon_2 \epsilon_3 \epsilon_4 \int_{t_1,t_2,t_3} B_{\epsilon_1,\epsilon_2} B_{\epsilon_3,\epsilon_4} A_2 \sin[v(t_1 - t_2 + v(t_3 - t_4))].$$

(E1)
where the symmetry between $t_3$ and $t_4$ is used to sum over $\mu = \pm$. Defining $F_{\mu,t_i} = \exp(i\hbar R_{\mu,t_i}/2) - 1$ Eq. (67) can be expressed as

$$A_2 = \left( F_{t_3,t_4}^{-1} + 1 \right) \left( F_{t_4,t_3}^{-1} + 1 \right) \left( F_{t_1,t_2}^{-1} + 1 \right) \left( F_{t_2,t_1}^{-1} + 1 \right)$$

$$\times \left( F_{t_1,t_3}^{-1} + 1 \right) \left( F_{t_3,t_1}^{-1} + 1 \right) \left( F_{t_2,t_2}^{-1} + 1 \right) \left( F_{t_4,t_4}^{-1} + 1 \right)$$

$$= \left( F_{t_1,t_3}^{-1} + 1 \right) \left( F_{t_3,t_1}^{-1} + 1 \right) \left( F_{t_2,t_2}^{-1} + 1 \right) \left( F_{t_4,t_4}^{-1} + 1 \right) \left( F_{t_1,t_3}^{-1} + 1 \right) \left( F_{t_4,t_4}^{-1} + 1 \right) \left( F_{t_2,t_2}^{-1} + 1 \right) \left( F_{t_4,t_4}^{-1} + 1 \right) \left( F_{t_1,t_3}^{-1} + 1 \right).$$

(E2)

In the last expression we used the retarded property of $F_{\mu,t_i}$ so that $F_{\mu,t_i} F_{\nu,t_i} = 0$. When transforming all functions to their frequency domain

$$\delta E^{(2)} = \frac{1}{\hbar \omega} \int d\omega d\omega^\prime \left[ \left( B_{\omega,\omega^\prime} + B_{\omega^\prime,\omega} \right) \left[ B_{\omega,\omega^\prime} - B_{\omega^\prime,\omega} \right] \right] \left[ \sum_{\mu} \epsilon_\mu \epsilon_\mu^\prime \epsilon_\mu^\prime \right] (F_{\mu,\omega} - (F_{\mu,\omega^\prime})^*) (F_{\mu,\omega^\prime}) (F_{\mu,\omega}^*) (F_{\mu,\omega^\prime})^*) \delta(\omega - \omega^\prime) \delta(\omega^\prime - \omega).$$

(E3)

We notice that the function $K(\omega_\mu,\omega_\nu)$ can have poles at $\omega_\mu, \omega_\nu = i \delta$ leading to a logarithmic divergence term for either a $O(\omega^{-1})$ term with the antisymmetric expression

$$\int d\omega \left( B_{\omega,\omega^\prime} + B_{\omega^\prime,\omega} \right) \frac{1}{\omega - i \delta} = -2 \int_0^\infty B \sin(\omega t) dt = \frac{2\hbar \eta}{\pi} \ln(\omega) + O(\omega),$$

(E5)

or for $O(\omega^{-2})$ terms with the symmetric expression

$$\int d\omega \left( B_{\omega,\omega^\prime} + B_{\omega^\prime,\omega} \right) \frac{1}{(\omega - i \delta)^2} = -2 \int_0^\infty \tau B \cos(\omega t) dt = \frac{2\hbar \eta}{\pi} \ln(\omega) + O(\omega),$$

(E6)

where $\delta = 0$. Note that the Fourier transform of $1/(\omega - i \delta)$ is $e^{-i\delta\Theta(\tau)}$ while that of $1/(\omega - i \delta)^2$ is $e^{-2i\delta\Theta(\tau)}$. We keep here only the long-time divergence, controlled by $\ln \omega$. Keeping also short-time divergences would eventually replace $\ln \omega \rightarrow \ln(\omega)$ with $\omega_\mu = \eta / m$. Equations (E5) and (E6) show that $\ln^2(\omega)$ terms arise from either a $1/(\omega_\mu^2 \omega_\nu)$ or $1/(\omega_\mu \omega_\nu^2)$ terms in $K(\omega_\mu,\omega_\nu)$.

We use the retarded property of $F_{\mu,t_i}$ for $F_{\mu,t_i}$, $\Theta(\tau)$ and expand the function in powers of $h/\eta$

$$F_{\omega}^{\mu} = e^{i\hbar \mu/(2\eta)} \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i\hbar \epsilon}{2\eta} \right)^n \frac{i}{\omega + i\eta \omega + i \delta}.$$

(E7)

Each of the six factors takes the form

$$F_{\omega}^{\mu} + F_{-\omega}^{\mu} + 2\pi \delta(\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i\hbar \epsilon}{2\eta} \right)^n \left\{ \frac{i e^{\mu \epsilon/(2\eta)}}{\omega + i\eta \omega + i \delta} + \frac{i e^{-\mu \epsilon/(2\eta)}}{-\omega + i\eta \omega + i \delta} \right\} \left( -\frac{1}{\omega^2} \right)^3,$$

(E8)

where the delta function cancels with the last terms of the $F_{\omega}$. We note that $\ln \omega$ terms arise from terms with at least one vanishing $n_j$, leading to a pole. For that particular $n_j$ the pole has a coefficient $\exp[i\epsilon_j \hbar/(2\eta)] - \exp[i\epsilon_j \hbar/(2\eta)]$ that vanishes when $\hbar/(2\eta) = \pi \times \text{integer}$. Hence all terms of $\delta E^{(2)}$ have at least one periodic factor of $\sin(\hbar/(2\eta))$.

The triple-frequency integral Eq. (E4) with the substitution (E8) has 24 terms with all three poles in either $\omega_\mu$ or $\omega_\nu$. Solving the triple integral and the $\epsilon_j$ summations we find

$$K(\omega_\mu,\omega_\nu) = \sum \frac{1}{n_1 n_2 n_3 n_4 n_5 n_6} \left( -\frac{i\hbar \epsilon}{2\eta} \right)^n \frac{2}{\omega^3} \left\{ \frac{(-1)^{n_2} (-1)^{n_1}}{(n_2 + n_3 + n_4 + n_5 + n_6 + i \eta \omega_\mu)} \right.$$
At this stage the \( \ln^2 v \) term can be simply identified, since this term needs poles in both \( \omega_a \) and \( \omega_b \). The only such term which has the form \( [(\omega_a - i \delta)(\omega_b - i \delta)]^{-1} \) is the term where \( n_1 = n_2 = \cdots = n_6 = 0 \); no other term has a zero-frequency divergence at both \( \omega_a \) and \( \omega_b \). For this term we get

\[
K_0(\omega_a, \omega_b) = \frac{16 \sin^2 [\hbar/(2 \eta)] \sin (\hbar/\eta)}{(\omega_a - i \delta \omega_a)^2 (\omega_b - i \delta \omega_b)}. \tag{E10}
\]

And the full expression from Eq. (E1), using Eqs. (E5) and (E6), is then

\[
\delta E^{(2)} = \frac{16}{16 \hbar^2} \frac{4 \eta^2}{\pi^2} v \ln^2(v) \sin^2 \frac{\hbar}{2 \eta} \sin \frac{\hbar}{\eta} + O(\ln v) = \frac{4 \eta^2}{\pi^2 \hbar} \sin^2 \frac{\hbar}{2 \eta} \sin h \ln^2(v) + O(\ln v). \tag{E11}
\]

This coefficient of the \( v \ln^2(v) \) term agrees with that of Eq. (70).

We consider next some of the terms that contribute to the \( \ln v \) coefficient. From Eq. (E5) we know that only terms with a single pole [i.e., either \( 1/(\omega_a - i \delta) \) or \( 1/(\omega_b - i \delta) \)] contribute. We define an expansion

\[
K(\omega_a, \omega_b) = K_0(\omega_a, \omega_b) + \sum_{\tilde{n} = 1}^{+\infty} \left( - \frac{i \hbar}{2 \eta} \right)^{\tilde{n}} \kappa_{\tilde{n}}(\omega_a, \omega_b), \tag{E12}
\]

where \( \tilde{n} = \sum_{j=1}^{n} n_j \). Thus there are six terms for \( \tilde{n} = 1 \), 21 terms for \( \tilde{n} = 2 \), and 56 terms for \( \tilde{n} = 3 \). Due to the \( \omega_a, \omega_b \) symmetry we define

\[
\kappa_{\tilde{n}}(\omega) = \lim_{\omega_a \to 0} \omega_a \kappa_{\tilde{n}}(\omega_a, \omega) + \lim_{\omega_b \to 0} \omega_b \kappa_{\tilde{n}}(\omega_b, \omega), \tag{E13}
\]

so that one integration gives a \( \ln v \) while the other gives its coefficient in the form

\[
\delta E^{(2)} = \frac{4 \eta^2}{\pi^2 \hbar} \sin^2 \frac{\hbar}{2 \eta} \sin \frac{\hbar}{\eta} v \ln^2(v) + \frac{\eta}{2 \omega_c^2 \hbar^2} \sum_{\tilde{n} = 1}^{+\infty} \left( - \frac{i \hbar}{2 \eta} \right)^{\tilde{n}} \int_{\omega} B_{\tilde{n}}(\omega) v \ln(v) + O(1). \tag{E14}
\]

For the first few terms we find

\[
\kappa_1(\omega) = \frac{P_1 + P_3 \cos \frac{\hbar}{\eta}}{P_4} \sin^2 \frac{\hbar}{2 \eta}, \quad \kappa_2(\omega) = \frac{P_4 + P_3 \cos \frac{\hbar}{\eta}}{P_6} \sin \frac{\hbar}{2 \eta}, \quad \kappa_3(\omega) = \frac{P_3 + P_8 \cos \frac{\hbar}{\eta}}{P_{10}} \sin^2 \frac{\hbar}{2 \eta}, \tag{E15}
\]

where \( P_I \) is a polynomial of \( \omega/\omega_c \) of degree \( I \). The result is consistent with having at least one factor of \( \sin[\hbar/(2 \eta)] \), as shown above in general.

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