Non-linear dynamics project
Guided by prof. Y. Zarmi

Stochastic resonance
Preface

Stochastic resonance (SR) provides a intriguing example of a noise-induced transition in a nonlinear system driven by an information signal and noise simultaneously. In the regime of SR some characteristics of the information signal (amplification factor, signal-to-noise ratio, the degrees of coherence and of order, etc.) at the output of the system are significantly improved at a certain optimal noise level. SR is realized only in nonlinear systems for which a noise-intensity-controlled characteristic time becomes available.

In my work I will provide a heuristic and systematic derivation of theoretical description of SR by reviewing scientific articles and I also represent some curious examples of its application.

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Contents

Introduction  4

1. Characteristics of SR by bistable system  6

2. Theoretical approaches  9

3. Two-state theory  12

4. Linear response  15

5. Application of SR in biological system  17

References
Introduction.

The word “noise” in ordinary consciousness is associated with the term ‘hindrance’. It was traditionally considered that the presence of noise can only make the operation of any system worse. Recently it has been established that the presence of noise sources in nonlinear dynamical systems can induce completely new regimes that cannot be realized without noise, for example, noise-induced self-sustained oscillations [5]. These effects were called noise-induced transitions [6]. The variety and complexity of the transitions in nonlinear dynamical systems produced the following questions, quite surprising until recently: does noise always disorder a system's behavior, or are there cases when noise enhances the degree of order in a system or evokes improvement of its performance? Recent studies have convincingly shown that in nonlinear systems noise can induce new, more ordered, regimes, lead to the formation of more regular structures or increase the degree of coherence, cause the amplification of weak signals and growth of their signal-to-noise ratio. In other words, noise can play a constructive role, enhancing the degree of order in a system. Stochastic resonance (SR) is one of the most shining and relatively simple examples of this type of nontrivial behavior of nonlinear systems under the influence of noise. The notion of SR determines a group of phenomena where in the response of a nonlinear system to a weak input signal can be significantly increased with appropriate tuning of the noise intensity. At the same time, the integral characteristics of the process at the output of the system, such as the spectral power amplification (SPA), the signal-to-noise ratio (SNR) or input-output cross-correlation measures have a well-marked maximum at a certain optimal noise level. Besides, an entropy-based measure of disorder attains a minimum, showing the increase of noise-induced order.

The term ‘stochastic resonance' was introduced by Benzi, Sutera and Vulpiani [7 - 9] in 1981 -1982, when they were exploring a model of a bistable oscillator proposed for explanation of the periodic recurrences of the Earth's ice ages.
The model described the motion of a particle subject to large friction in a symmetric double-well potential, driven by a periodic force. Two stable states of the system in Benzi, Sutera and Vulpiani model represented by the earth states -covered with ice and without ice, which corresponding optimal climate. (covered with ice earth state would reflect significant percent solar light which lead to global decreasing of temperature and prevent ice melting and vice versa after if ice began to melt earth will gradually absorb more light and it will be prevent to form new glacier).

The periodic forces in bistable model (figure 1) referred to the oscillations of the eccentricity (now it is equal 0.0167) of the Earth's orbit with periodicity about 100000 years and tilt change from 0° to 90° with periodicity about 41000 years (now it is 23.5°). Estimations have shown that the oscillations of solar energy is ~ 0.1% per year, so the actual amplitude of the periodic force is far too small to cause the system to switch from one state to another one. The possibility of switchings was achieved by the introduction of additional random force, (or noise), such random force can be season oscillation of temperature which induced transitions from one state to another over the potential barrier of the system.

In 1983, SR was studied experimentally in the Schmitt trigger system where the SNR was first used to describe the phenomenon [10]. In has been shown that the SNR at the output of the Schmitt trigger subjected to a weak periodic signal and noise increases with increasing noise.
intensity, passes through a maximum and then decreases. By this means there is an optimal noise level at which the periodic component of the signal is maximized.

Thereafter the effect of SR has been found and studied in a variety of physical systems, namely, in a ring laser [11], in magnetic systems [12], in passive optical bistable systems [13], in systems with electronic paramagnetic resonance [14], in experiments with Brownian Particles [15], in experiments with magnetoelastic ribbons [16], in a tunnel diode [17], in superconducting quantum interference devices (SQUIDs) [18], and in ferromagnetics and ferroelectrics [19 - 21]. SR has equally been observed in chemical systems [22 - 24] and even in social models [25].

1. Characteristics of SR by bistable system

We start considering SR from bistable system (system with two stable states), and simple physical example for such system is Brownian particle in a symmetric double-well potential that described by $U(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ (figure 2) in the absence of periodic modulation ($A=0$), the system possesses two characteristic time scales. The first is defined by random walks of the particle in the vicinity of one of the equilibrium positions (the interawell or local dynamics). The second time scale characterizes the mean time of barrier crossing and refers to the global dynamics of noise-induced transitions between the potential wells. Now we are adding periodic force $A\sin(\Omega t)$ that leads to a periodic modulation of both the barrier height $\Delta U \cdot \Delta U_0 + A\sin(\Omega t)$ and the probability of a switching event. Note that the amplitude of the periodic force $A$ is assumed to be small in the sense that switchings between the potential wells are excluded in the absence of noise. Although the periodic force is too weak to let the particle roll periodically from one potential well into the other one, noise induced hopping between the potential wells can become synchronized with the weak periodic forcing. This statistical synchronization takes place when the average time between jumps across barrier $T_K(D)=1/r_K$ (where $r_K$ mean rate (or frequency) of escape from a
A metastable state, known as the Kramer’s rate and defined by the Arrhenius law

\[ r_k = \frac{1}{2\pi} \sqrt{u^\prime(x_{\text{max}})u^\prime(x_{\text{max}})} \exp \left( -\frac{\Delta U}{D} \right) \]

of two noise-induced interwell transitions is comparable with half the period \( T_\Omega \) of the periodic force. This yields the **time-scale matching condition** for stochastic resonance is \( T_K(D) = T_\Omega/2 \) or other words when a modulation signal is added the probability density \( p(\tau) \) becomes structured and contains a series of Gaussian-like peaks centered at \( \tau = nT_\Omega/2 \), \( n = 1, 3, 5... \) (where \( \tau \) is residence time for one the potential wells).

Now let examine some properties of SR (figure 3) versus different noise intensities when (a) noise intensity is low particle will be located one of the well and periodic signal would not be appear, by increasing noise to optimal value, the particle will be in resonance state and will jump from one well to another synchronically and periodic component of response will be same as external force, and finally after we increase noise intensity more the periodic response will be decrease and motion of particle will become more chaotic.

In figure 4 showed typical depending system response versus noise intensity where we can clearly see that exist optimal noise intensity for which response of system is maximal.
There is also other way to describe SR. Instead of taking the ensemble average of the system response, it sometimes can be more convenient to extract the relevant phase-averaged power spectral density $S(\omega)$ defined as

$$S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \tau} \left\langle \{x(t + \tau)x(t)\} \right\rangle d\tau$$

where the outer brackets denote the average over the input initial phase and inner brackets denote the ensemble average over the realizations of the noise. In Fig.5 display a typical example of $S(\nu) (\nu=2 \pi \nu)$ for the bistable system. Qualitatively, $S(\omega)$ may be described as the superposition of a background power spectral density $S_N(\omega)$ and a structure of delta spikes centered at $\omega=(2n+1)\Omega$ with $n=0,\pm1,\pm2$.

![Figure 5](image)

Summarizing said all above we saw that there are three basic requirements for SR: source of background noise, weak coherent input and characteristic "barrier" or threshold.

And before starting theoretical approaches for SR it necessary underline that we will try to characterize our system by two parameters:

$$\text{SNR} = \frac{P_S}{P_N} \quad \text{and} \quad \eta - \text{SPA} = \left[ \frac{A_{out}}{A_{in}} \right]^2$$

(spectral power amplification)
2. Theoretical approaches

Now we are starting to consider SR by model of an overdamped bistable oscillator which can be described by canonic equation of a Brownian particle motion in dimensionless variables in a double-well potential \( U(x) = -\frac{1}{2} x^2 + \frac{1}{4} x^4 \) driven by white noise \( \xi(t) \) with the intensity \( D \) and periodic force \( f(t) = A \cos(\Omega t + \phi) \), so our eq. is

\[
\dot{x} = x - x^3 + A \cos(\Omega t + \phi) + \sqrt{2D} \cdot \xi(t) \quad \text{eq (2.1)}
\]

In order to find solution for this equation we build the corresponding Fokker-Planck equation (FPE) for the probability density \( p(x,t,\phi) \) as follows

\[
\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left\{ x - x^3 + A \cos(\Omega t + \phi) \right\} p + D \frac{\partial^2 p}{\partial x^2} \quad \text{eq (2.2)}
\]

In the general case it is impossible to find an exact solution of the Fokker-Planck equation for a time-dependent probability density and for this reason we cannot calculate exactly the correlation functions and power spectral densities. On the other hand, with the periodic force taken into account, certain additional difficulties arise due to the inhomogeneity of the corresponding stochastic process in time but anyway we can get approximate solution we can rewrite FPE in operator form \( \frac{\partial p}{\partial t} = \left[ L_0' + L_{\text{ext}}(t) \right] p \) where

\[
L_0' = -\frac{\partial}{\partial x} \left( x - x^3 \right) + D \frac{\partial}{\partial x^2} \quad \text{is the unperturbed Fokker-Planck operator (A=0)}
\]

\[
L_{\text{ext}}(t) = A \cos(\Omega t + \phi) \quad \text{denotes the periodic gradient-type perturbation}
\]

Now we try to use Floquet approach (Floquet, 1883; Magnus and Winkler, 1979) which applied to the inertial, as well as the overdamped Brownian dynamics and corresponding partial differential equation. It describes a nonstationary Markovian process where the symmetry under time translation is retained in a discrete manner only. Since the Fokker-
Planck operators are invariant under the discrete time translations $t \to t + T_\Omega$, where $T_\Omega = \frac{2\pi}{\Omega}$ denotes the modulation period they correspond $\mathcal{L}(t) = \mathcal{L}(t + T_\Omega)$

on the multidimensional space of state vectors $X(t) = (x(t); \nu(t) = \dot{x}(t); \ldots)$ and it can be found that the relevant Floquet solutions are functions of the type $p(X,t,\varphi) = \exp(-\mu t) p_\mu (X,t,\varphi)$ with $\mu$ is Floquet eigenvalue and $p_\mu$ periodic Floquet modes $p(X,t,\varphi) = p_\mu (X,t + T_\Omega,\varphi)$

so we can re-write FPE $\left( \mathcal{L}(t) - \frac{\partial}{\partial t} \right) p_\mu (X,t,\varphi) = -\mu p_\mu (X,t,\varphi)$

Here the Floquet modes $p_\mu$ are elements of the product space $L_1 \oplus T_\Omega$ where $L_1(X)$ is the linear space of the functions that are integrable over the state space. In view of the identity

now let’s introduce the set of Floquet modes of the adjoint operator $\mathcal{L}^\dagger(t)$, that is

$$\left( \mathcal{L}^\dagger(t) - \frac{\partial}{\partial t} \right) p^\dagger_\mu (X,t,\varphi) = -\mu p^\dagger_\mu (X,t,\varphi) \quad \text{eq (2.3)}$$

Here the sets $p_\mu$ and $p^\dagger_\mu$ are bi-orthogonal, obeying the equal-time normalization condition

$$\frac{1}{T_\Omega} \int_{0}^{T_\Omega} dt \int dX p_\mu (X,t,\varphi) p^\dagger_\mu (X,t,\varphi) = \delta_{n,m} \quad \text{eq (2.4)}$$

Eqs. (2.3) and (2.4) allow for a spectral representation of the time inhomogeneous conditional probability $P(X,t;Y,s)$: With $t>s$ we find

$$P(X,t|Y,s) = \left[ \sum_{n=0}^{\infty} p_\mu (X,t,\varphi) p^\dagger_\mu (Y,s,\varphi) \right] e^{-\mu_n (t-s)} = P(X,t+T_\Omega|Y,s+T_\Omega) \quad \text{eq (2.5)}$$

With all real parts $\Re[\mu_n] > 0$ for $n > 0$, the limit $s \to -\infty$ of Eq. (2.5) yields the ergodic, time-periodic probability $p_{as}(X,t,\varphi) = p_{\mu=0}(X,t,\varphi)$

where the asymptotic probability $p_{as}(X,t,\varphi)$ can be expanded into a Fourier series, i.e.,

$$p_{as}(X,t,\varphi) = \sum_{n=-\infty}^{\infty} a_n(X) \exp\left[ i n (\Omega t + \varphi) \right] \quad \text{eq (2.6)}$$

With the arbitrary initial phase being distributed uniformly, i.e., with the probability density for $\varphi$
given by \( w(\varphi) = (2\pi)^{-1} \), the time average of Eq. (2.6) equals the phase average. Hence

\[
\overline{p}_{\text{as}} = \frac{1}{2\pi} \int_{0}^{2\pi} d\varphi p_{\text{as}}(X, t, \varphi) = \frac{1}{T_\Omega} \int_{0}^{T_\Omega} p_{\mu=0}(X, t, \varphi) dt
\]
eq (2.7)

At this stage it is worth pointing out a peculiarity of all periodically driven stochastic systems:

With \( \theta = \Omega t + \varphi \) we could as well embed a periodic \( N \)-dimensional Fokker-Planck equation into a Markovian \( (N+1) \)-dimensional, time-homogeneous Fokker-Planck equation by noting that \( \dot{\theta} = \Omega \). With the corresponding stationary probability \( p_{\text{as}}(x, \theta) \) not explicitly time dependent.

Particular importance for stochastic resonance is the asymptotic expectation value it is also periodic in time and thus admits the Fourier series representation

\[
\langle x \rangle_{\text{as}} = \langle X(t) \rangle_{Y(t) \to -\infty} = \sum_{n=-\infty}^{\infty} M_n(\Omega, A) \exp\left[ in(\Omega t + \varphi) \right]
\]
eq (2.8)

The complex-valued amplitudes \( M_n(\Omega, A, D) \) depend nonlinearly on both the forcing frequency \( \Omega \) and the modulation amplitude \( A \). Nonlinear contributions to the stochastic resonance observables, both for \( M_1 \) and higher-order harmonics with \( n > 1 \) have been evaluated numerically by Jung and Hanggi (1989, 1991) by implementing the Floquet approach for the Fokker-Planck equation of the overdamped driven quartic double-well potential.

The spectral amplification

\[
\eta = \frac{P_x}{P_A} = \left( \frac{\overline{X}}{A} \right)^2
\]

of the integrated power in the time-averaged power spectral density can be expressed in terms of \( M_1 \), i.e.

\[
\eta = \left( \frac{2|M_1|}{A} \right)^2
\]

**Figure 6**

The spectral amplification \( \eta \) versus the noise intensity \( D \) at a fixed modulation frequency \( \Omega = 0.1 \) is depicted for four values of the driving amplitude \( A \). The result of the linear response approximation is depicted by the dotted line. From Jung and Hanggi (1991).
Analytical expressions for the spectral power amplification and the signal-to-noise ratio can be derived via some approximations. One of the main is a weak signal approximation when the response can be considered as linear. Other approximations is two-state theory where we have to impose some restrictions on the signal frequency.

3. Two-state model

Two-state model describes a symmetric bistable system with a discrete state variable \( x(t) = \pm x_m \) and \( n_\pm(t) \) be the probabilities of residing the system in the corresponding state, that satisfying the normalization conditions \( n_\pm(t) + n_\mp(t) = 1 \). And now we define probability densities of switching from one state to another by \( W_\pm(t) \)

So we can arrive to eq.

\[
\dot{n}_\pm = -\left[ W_\pm(t) + W_\mp(t) \right] n_\pm + W_\pm(t)
\] (3.1)

which can be solved analytically

\[
W_\pm(t) = r_K e^{-\frac{A_{\text{eff}} \cos(\Omega t)}{D}}
\]

was proposed by McNamara and Wiesenfeld (1989) proposed to use periodically modulated escape rates of the Arrhenius type) where without external force \( (A = 0) \), the probability densities of switching coincide with the Kramer’s rate \( r_K \) and are independent of time and solution of the probability equation is given by

\[
n_\pm = g(t) \left[ n_\pm(t_0) + \int_{t_0}^{t} W_\mp(t) g^{-1}(t) \, dt \right] \text{where } g(t) = \exp \left( -\int_{t_0}^{t} W_\mp(t) + W_\mp(t) \, dt \right)
\] (3.2)

The master equation (3.1) adequately describes the dynamics of the bistable overdamped oscillator when the signal changes slowly enough so that relaxation processes in the system go much more faster than the external force changes. The equations for conditional probabilities \( n_\pm(t| x_0, t_0) \), which we need to calculate the autocorrelation function, have the same form as (3.1). A solution of the linear equation (3.1) can be obtained
without any difficulty for the case of a weak signal, i.e. $A x_m \ll D A$. In this case one can
expand the switching rate in a Taylor power series and retain the linear terms in the signal
amplitude only. As a result, the expression for the conditional probability reads as

$$n_s(t|x_0, t_0) = \frac{1}{2} \left\{ e^{-2r_K(t-t_0)} \left( 2\delta x_0, x_m - 1 - \frac{2r_K A x_m \cos(\Omega t_0 + \psi)}{D \left( 4r_K^2 + \Omega^2 \right)} \right) + 1 + \frac{2r_K A x_m \cos(\Omega t + \psi)}{D \left( 4r_K^2 + \Omega^2 \right)} \right\}$$

where $\psi = -\arctan(\Omega / 2r_K)$. Knowledge of the conditional probabilities allows us to calculate
any statistical characteristics of the process. The mean value characterizing the system's response and the spectral density that is necessary for estimating the SNR are of great interest. The conditional probability density is determined as

$$p(x,t|x_0, t_0) = n_s \delta(x-x_m) + n \delta(x+x_m)$$

and where the mean value is $\langle x(t)|x_0, t_0 \rangle = \int x p(x,t|x_0, t_0) dx$. We are interested in the asymptotic limit $\langle x(t) \rangle = \lim_{t_0 \to -\infty} \langle x(t)|x_0, t_0 \rangle$ for which from (3.2) we obtain $\langle x(t) \rangle = A_1(D) \cos[\Omega t + \psi(D)]$

where the amplitude $A_1(D)$ and the phase shift $\psi(D)$ are given by the following expressions

$$A_1 = \frac{2A x^2 r_K}{\left( 4r_K^2 + \Omega^2 \right)^{3/2}} \quad \text{and} \quad \psi = -\arctan(\Omega / 2r_K)$$

Knowing the signal amplitude at the output we can determine the SPA as

$$\eta = \frac{4r_K x_m^4}{D^2 \left( 4r_K^2 + \Omega^2 \right)}$$

Similarly we can find the autocorrelation function

$$\langle x(t+\tau|x_0, t_0) \rangle = \int x y p(x,t+\tau|y,t) p(y,t|x_0, t_0) dxdy$$

and its asymptotic limit for $t_0 \to -\infty$. However, by virtue of periodic modulation the autocorrelation function depends not only on the time shift $t$ but also periodically on the time $\tau$.

In order to calculate the spectral density one needs to perform additional averaging over the period of the external force. Such a procedure is equivalent to averaging over an ensemble of the initial random phases and corresponds to experimental
methods for measuring the spectral densities and the correlation functions. The expression for the spectral density \( S(\omega) \) has the form and contains two components, namely, a periodic one represented by a delta-function with an appropriate weight, and a noisy component \( S_N(\omega) \).

\[
S(\omega) = S_N(\omega) + \frac{\pi}{2} \cdot A_\omega \cdot \left[ \delta(\omega - \Omega) + \delta(\omega + \Omega) \right]
\]

where \( S_N(\omega) = \frac{4r_Kx^2_m}{4r^2_K + \omega^2} \left[ 1 - \frac{A_\omega}{2x^2_m} \right] \)

As seen from the last expression, the noise background is represented by the sum of the unperturbed spectrum (for \( A=0 \)) \( S^{(0)}_N(\omega) = \frac{4r_Kx^2_m}{4r^2_K + \omega^2} \) and a certain additional term of order \( A^2 \): \( S(\omega) = S_N(\omega) + O(A^2) \).. The spectral density of the unperturbed system is determined by the noise intensity \( D \) that is involved in the expression for the Kramer’s rate \( r_K \). The explicit expression for the unperturbed spectral density is given by formula (4.3). For small \( D \), the Kramers rate is small and the spectrum is centered in the low-frequency range. With increasing noise intensity the Kramer’s rate rises exponentially and the spectrum becomes more uniform. The appearance of the additional term reducing the noise background is explained by the fact that for the two-state model the energy of the output process constitutes \( 2\pi x^2_m \) and does not depend on the signal amplitude and frequency. The signal-to-noise ratio for the two-state model reads as

\[
\text{SNR} = \pi \left( \frac{Ax_m}{D} \right)^2 r_K + O(A^4)
\]
4. Linear-response theory

the linear-response concept is based on perturbation theory. According to this theory, the response of a non-linear stochastic system \( \langle x(t) \rangle \) to a weak external force \( f(t) \) in the asymptotic limit of large times is determined by the integral relation [28]

\[
\langle x(t) \rangle = \langle x \rangle_s + \int_{-\infty}^{\infty} \chi(t-\tau,D)f(\tau)d\tau
\]

where \( \langle x \rangle_s \) is the mean value of the unperturbed state variable \( \left[ f(t) = 0 \right] \) of the system, and \( f(t) \) is the external disturbing force. Without lack of generality, we set \( \langle x \rangle_s = 0 \) for simplicity. This condition holds for symmetric systems and, in particular, for the base model (2.1). The function \( \chi(t) \) is called the response function and for systems which in the absence of perturbations are in thermodynamic equilibrium, is connected with the correlation functions of the unperturbed system via fluctuation-dissipation relations [29].

we will discuss the application of linear response theory (LRT) to SR using the overdamped bistable system (2.1).

The fluctuation-dissipation theorem connecting the response function \( \chi(t) \) and autocorrelation function \( K^{(0)}_{xx}(t) \) of the unperturbed system by the form

\[
\chi(t) = -\frac{l(t)}{D} \frac{d}{dt} K^{(0)}_{xx}(t)
\]

(4.1)

where \( l(t) \) is the Heaviside function. In order to calculate the characteristics of the system response in the framework of LRT we need to know the statistical properties of the system in its unperturbed equilibrium or stationary state. For a harmonic force the system response is expressed through the susceptibility \( |\chi(\omega)| \) which is a Fourier transform of the response function

\[
\langle x(t) \rangle = A|\chi(\omega)|\cos(\Omega t + \psi)
\]

where phase shift given by \( \psi = -\arctan \frac{\text{Im} \chi(\Omega)}{\text{Re} \chi(\Omega)} \)

the SPA defined as \( \eta = |\chi(\Omega)|^2 \) and \( \text{SNR} = \frac{\pi A^2 |\chi(\Omega)|^2}{S^{(0)}_{xx}(\Omega)} \)

(4.2)
It is impossible to get an exact expression for the autocorrelation function $K_{xx}^{(0)}(t, D)$

However, there are few approaches to its approximate evaluation. The most precise approach is based on the expansion of the Fokker - Planck operator in terms of eigenfunctions [30]. The correlation function can be represented therewith as a series of $g_j \exp(-\lambda_j t)$, where $\lambda_j$ are the eigenvalues of the Fokker-Planck operator, and $g_j$ are the coefficients which are computed by averaging the corresponding eigenfunctions over the unperturbed equilibrium distribution.

In the simplest case, when calculating the correlation function one may take into account a least nonvanishing eigenvalue $\lambda_m$ that is related to the Kramer’s rate of the escape from a potential well: $\lambda_m = 2r_K = \frac{\sqrt{2}}{\pi} \exp\left(-\frac{1}{4D}\right)$

In this case the expressions for the correlation function and the spectral density of the unperturbed system are

\[
K_{xx}^{(0)}(\tau, D) \approx \left\langle x^2 \right\rangle \exp(-\lambda_m \tau)
\]

\[
S_{xx}^{(0)}(\Omega) \approx \frac{\left\langle x^2 \right\rangle_{st}}{\lambda_m^2 + \Omega^2}
\]

where $\left\langle x^2 \right\rangle_{st}$ is the stationary value of the second cumulant of the unperturbed system

\[
\left\langle x^2 \right\rangle_{st} = C \cdot \int_{-\infty}^{\infty} x^2 \exp\left[\frac{1}{D} \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\right] dx = \int_{-\infty}^{\infty} x^2 p_{xx}(x) dx
\]

and C is the normalization constant for the stationary probability density. Such an approximation corresponds to the two-state approach and cannot be applied for small noise intensities $D \ll 1$ and exposure to high-frequency external disturbance.
5. Application of SR in biological system

The prominent role of the stochastic resonance phenomenon is that it can be used to boost weak signals embedded in a noisy environment. Applications of SR to sensory biology are most interesting and, perhaps, most important. There are several reasons to believe that living organisms have adapted with evolution to use the inevitable internal noise and noisy environment for optimal detection and extraction of useful information. In particular, it has been demonstrated by research of sensory processes in the hydrodynamically sensitive mechanoreceptors of the crayfish [26, 27].

The mechanoreceptor system of the crayfish is located in its tailfan and is illustrated in Fig. 6. The tailfan has approximately 250 long hairs which are connected to the interneurons within the ganglion by sensory neurons collected into nine nerve roots. The tailfan has the 6th, or terminal, ganglion with its pair of photoreceptor cells. As seen from Fig. 7, neural impulses can be registered at the sensory neurons, or on the photoreceptor output neuron. In order to record the neuron signals, the microelectrodes were surgically introduced into desired parts of the crayfish receptor system. The hairs were stimulated by relative fluid motions in the directions shown in Fig. 7. These motions are typically sinusoidal, of amplitude 10 to 100 nanometers, at frequencies from 5 to 100 Hz, and velocities 100 to 1000 microns per second. The responses of a photoreceptor cell can be studied in the presence of both hydrodynamical stimulation and light beam of uniform intensity on the photoreceptive area. Environmental noise was used as the random source. The experimental protocols necessary for this research have been well-developed [26]. Measurements of the SNR of the crayfish sensory neuron as a function of the external noise intensity have demonstrated the effect of SR.
The experimental results are shown by the squares in Fig. 8, theory is exhibited by the solid curve and a FitzHugh-Nagumo simulation is depicted by the diamonds. These results suggest that the mechanoreceptor system provides signal detection in environmental noise of an optimal intensity. Moreover, the signals best detected are sinusoidal in the frequency range around 10 Hz. This is the frequency characteristic of water vibrations induced by swimming fish. Such water waves travel faster than the fish itself thus providing an early warning of the crayfish.
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