Lecture Notes in Physics
Introduction to Plasma Physics

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Chapter 1

Basic definitions and parameters

In this chapter we learn in what conditions a new state of matter - plasma - appears.

1.1 What is plasma

Plasma is usually said to be a gas of charged particles. Taken as it is, this definition is not especially useful and, in many cases, proves to be wrong. Yet, two basic necessary (but not sufficient) properties of the plasma are: a) presence of freely moving charged particles, and b) large number of these particles. Plasma does not have to consists of charged particles only, neutrals may be present as well, and their relative number would affect the features of the system. For the time being, we, however, shall concentrate on the charged component only.

Large number of charged particles means that we expect that statistical behavior of the system is essential to warrant assigning it a new name. How large should it be ? Typical concentrations of ideal gases at normal conditions are $n \sim 10^{19}\text{cm}^{-3}$. Typical concentrations of protons in the near Earth space are $n \sim 1 - 10\text{cm}^{-3}$. Thus, ionizing only a tiny fraction of the air we should get a charged particle gas, which is more dense than what we have in space (which is by every lab standard a perfect vacuum). Yet we say that the whole space in the solar system is filled with a plasma. So how come that so low density still justifies using a new name, which apparently implies new features ?

A part of the answer is the properties of the interaction. Neutrals as well as charged particles interact by means of electromagnetic interactions. However, the forces between neutrals are short-range force, so that in most cases we can consider two neutral atoms not affecting one another until they collide. On the other hand, each *charged* particle produces a long-range field (like Coulomb field), which can affect many particles at a distance. In order to get a slightly deeper insight into the significance of the long-range fields, let us consider a gas of immobile (for simplicity) electrons, uniformly distributed inside an infinite cone, and try to answer the question: which electrons affect more the one which is in the cone vertex ? Roughly speaking, the Coulomb force acting on the chosen electron from another one which is at a distance r, is inversely proportional to the distance squared, $f_r \sim 1/r^2$. Since the number of electrons which are at this distance, $N_r \propto r^2$, the total force, $N_r f_r \sim r^0$, is distance independent, which means that that electrons which are very far away are of equal importance as the electrons which are very close. In other words, the chosen *probe* experiences influence from a large number of particles or the whole system. This brings us to the first hint: *collective* effects may be important for a charged particle gas to be able to be called plasma.

1.2 Debye shielding

In order to proceed further we should remember that, in addition to the density n, every gas has a temperature T, which is the measure of the random motion of the gas particles. Consider a gas of identical charged particles, each with the charge q. In order that this gas not disperser immediately we have to compensate the charge density nq with charges of the opposite sign, thus making the system *neutral*. More precisely, we have to neutralize locally, so that the positive charge *density* should balance the negative charge *density*. Now let us add a test charge Q which make slight imbalance. We are interested to know what would be the electric potential induced by this test charge. In the absence of plasma the answer is immediate: $\phi = Q/r$, where r is the distance from the charge Q. Presence of a large number of charged particles, which can move freely, changes the situation drastically. Indeed, it is immediately clear that the charges of the sign opposite to Q, are attracted to the test charge, while the charges of the same sign are repelled, so that there will appear an opposite charge density in the near vicinity of the test charge, which tends to neutralize this charge in some way. If the plasma particles were not randomly moving due to the temperature, they would simply stick to the test particle thus making it "neutral". Thermal motion does not allow them to remain all the time near Q, so that the neutralization cannot be expected to be complete. Nevertheless, some neutralization will occur, and we are going to study it quantitatively.

Before we proceed further we have to explain what electric field is affected. If we measure the electric field in the nearest vicinity of any particle, we would recover the single particle electric field (Coulomb potential for an immobile particle or Lienard-Viechert potentials for a moving charge), since the influence of other particles is weak. Moreover, since all particles move randomly, the electric field in any point in space will vary very rapidly with time. Taking into account that the number of plasma charges producing the electric field is large, we come to the conclusion that we are interested in the electric field which is averaged over time interval large enough relative to the typical time scale of the microscopic field variations, and over volume large enough to include large number of particles. In other words, we are interested in the statistically averaged, or *self-consistent* electric potential.

Establishing the statistical (or average) nature of the electric field around the test charge we are able now to use the Poisson equation

$$\Delta\phi = -4\pi\rho - 4\pi Q\delta(\boldsymbol{r}), \tag{1.1}$$

where the last term describes the test point charge in the coordinate origin, while ρ is the charge density of the plasma particles,

$$\rho = q(n - n_0). \tag{1.2}$$

Here n is the density of the freely moving charges in the presence of the test charge, while n_0 is their density in the absence of this charge. Assuming that the plasma is in thermodynamic equilibrium, we have to conclude that the charged particle are distributed according to the Boltzmann law

$$n = n_0 \exp(-U/T), \tag{1.3}$$

where $U = q\phi$. Strictly speaking, the potential in the Boltzmann law should be the local (non-averaged) potential, and averaging

$$\langle \exp(-q\phi/T)\rangle \neq \exp(-q\rangle\phi\rangle/T).$$

However, sufficiently far from the test charge, where $q\phi/T \ll 1$ we may Taylor expand

$$\langle \exp(-q\phi/T)\rangle = 1 - \langle \frac{q\phi}{T}\rangle$$

so that

$$\rho = -\frac{n_0 q^2}{T}\phi, \tag{1.4}$$

where now ϕ is the self-consistent potential we are looking for. Substituting into (1.1), one gets

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\phi = \frac{4\pi n_0 q^2}{T}\phi \tag{1.5}$$

for $r > 0$ and boundary conditions read $\phi \to Q/r$ when $r \to 0$, and $\phi \to 0$ when $r \to \infty$. The above equation can be rewritten as follows:

$$\frac{d^2}{dr^2}(r\phi) = \frac{1}{r_D^2}(r\phi), \tag{1.6}$$

where

$$r_D = \sqrt{T/4\pi n_0 q^2} \tag{1.7}$$

is called Debye radius. The solution (with the boundary conditions taken into account) is

$$\phi = \frac{Q}{r}\exp(-r/r_D). \tag{1.8}$$

We see that for $r \ll r_D$ the potential is almost not influenced by the plasma particles and is the Coulomb potential $\phi \approx Q/r$. However, at $r > r_D$ the potential decreases exponentially, that is, faster than any power. We say that the plasma charges effectively screen out the electric field of the test charge outside of the Debye sphere $r = r_D$. The phenomenon is called Debye screening or shielding, and is our first encounter with the collective features of the plasma. Indeed, the plasma particles act together, in a coordinated way, to reduce the influence of the externally introduced charge. It is clear that this effect can be observed only if Debye radius is substantially smaller than the size of the system, $r_D \ll L$. This is one of the necessary conditions for a gas of charged particle to become plasma.

It is worth reminding that the found potential is the potential averaged over spatial scales much large than the mean distance between the particles, and over times much larger than the typical time of the microscopic field variations. These variations (called fluctuations) can be observed and are rather important for plasmas' life. We won't discuss them in our course.

The two examples of the collective behavior of the plasma (Debye shielding and plasma oscillations) show one more important thing: the plasma particles are "connected" one to another via *self-consistent* electromagnetic forces. The self-consistent electromagnetic fields are the "glue" which makes the plasma particles behave in a coordinated way and this is what makes plasma different from other gases.

1.3 Plasma parameter

Since the derived screened potential should be produced in a statistical way by many charges, we must require that the number of particles inside the Debye sphere be large, $N_D \sim nr_D^3 \gg 1$. The parameter $g = 1/N_D$ is often called the plasm parameter. We see that the condition $g \ll 1$ is necessary to ensure that a gas of charged particles behave collectively, thus becoming plasma.

We can arrive at the same parameter in a different way. The average potential energy of the interaction between two charges of the plasma is $U \sim q^2/\bar{r}$, where $\bar{r}$ is the mean distance between the particles. The latter can be estimated from the condition that the there is exactly particle in the sphere with the radius $\bar{r}$: $n\bar{r}^3 \sim 1$, so that $\bar{r} \sim n^{-1/3}$. The average kinetic energy of a plasma particle is nothing but T, so that

$$\frac{U}{K} \sim \frac{q^2 n^{1/3}}{T} \sim \frac{1}{n^{2/3} r_D^2} = g^{2/3}. \tag{1.9}$$

If $g \ll 1$, as it should be for a plasma, then the average potential energy is substantially smaller than the average kinetic energy of a particle. In fact, we could expect that since in order that the charges be able to move freely, the interaction with other particles should not be too binding. If $U/K \ll 1$, the plasma is said to be ideal, otherwise it is non-ideal. We see that only ideal plasmas are plasma indeed, otherwise the substance is more like a charged fluid with typical liquid properties.

1.4 Plasma oscillations

In the analysis of the Debye screening the plasma was assumed to be in the equilibrium, that is, the plasma charges were not moving (except for the fast random motion which is averaged out). Thus, the screening is an example of the *static* collective behavior. Here we are going to study an example of the *dynamic* collective behavior.

Let us assume that the plasma consists of freely moving electrons and an immobile neutralizing background. Let the charge of the electron be q, mass m, and density n. Let us assume that, for some reason, all electrons, which were in the half-space $x > 0$, move to the distance d to the right, leaving a layer of the non-neutralized background with the charge density $\rho = -nq$ and width d. The electric field, produced by this layer on the electrons on *both edges* is $E = 2\pi\rho d = -2\pi nqd$ (for the electrons at the right edge) and $E = 2\pi\rho d = 2\pi nqd$ (for the electrons at the left edge). The force $F = qE = -2\pi nq^2 d$ accelerates the electrons at the right edge to the left, while the electrons at the left edge experience similar acceleration to the right. The relative acceleration of the electrons at the right and left edges would be $a = 2(qE/m) = -4\pi nq^2 d/m$. On the other hand, $a = \ddot{d}$, so that one has

$$\ddot{d} = -\omega_p^2 d, \qquad \omega_p^2 = 4\pi nq^2/m. \tag{1.10}$$

The derived equation describes oscillations with the *plasma frequency* ω_p. It should be emphasized that the motion is caused by the coordinated movement of many particles together and is thus a purely collective effect. In order to be able to observe these oscillations their period should be much smaller than the typical life time of the system.

1.5 Ionization degree

A plasma does not have to consist only of electrons, or only of electrons and protons. In other words, neutral particles may well be present. In fact, most laboratory plasmas are only partially ionized. They are obtained by breaking neutral atoms into positively charged ions and negatively charged electrons. The relative number of ions and atoms, n_i/n_a, is called the degree of ionization. In general, it depends very much on what is making ionization. However, in the simplest case of the thermodynamic equilibrium the ionization degree should depend only on the temperature. Indeed, the process if ionization-recombination, $a \leftrightarrow i + e$, is a special case of a chemical reaction (from the point of view of thermodynamics and statistical mechanics). Let I be the ionization potential, that is, the energy needed to separate electron from an atom. Then

$$\frac{n_i}{n_a} = \frac{g_{i+e}}{g_a} \exp(-I/T), \tag{1.11}$$

where $g_{i+e} = g_i g_e$ and g_a are the number of possible states for the ion+electron and atom, respectively. Usually $g_a \sim g_i \sim 1$. However, g_e is large. It can be calculated precisely but we shall make a simple estimate to illustrate the methods, which are widely used in plasma physics. The number of available states for an electron $g_e \sim \Delta V \Delta^3 p/h^3$, where ΔV is the volume available for one electron, $\Delta^3 p$ is the volume in the momenta space, and h is the Planck constant. It is obvious that $\Delta V \sim 1/n_e$. The volume in the momenta space can be estimated if we remember that the typical kinetic energy of a thermal particle, $p^2/m \sim T$, from which $\Delta^3 p \sim (Tm_e)^{3/2}$. Eventually,

$$g_e \sim \frac{(Tm_e)^{3/2}}{n_e h^3}, \tag{1.12}$$

and (1.11) takes the following form:

$$\frac{n_i n_e}{n_a} \sim \frac{(Tm_e)^{3/2}}{h^3} \exp(-I/T). \tag{1.13}$$

Let us proceed further by assuming that the ions are singly ionized, which gives $n_i = n_e$, and introduce the total density $n = n_a + n_i$ and the ionization degree $z = n_i/n$, then one has

$$\frac{z^2}{1-z} \sim \frac{(Tm_e)^{3/2}}{nh^3} \exp(-I/T). \tag{1.14}$$

When the density is low, the pre-exponential in (1.14) is large, and even for $T < I$ the ionization degree z may be close to unity, $1 - z \ll 1$. In this case we say that the plasma is fully ionized. The expression (1.14) in its precise form is called Saha formula.

1.6 Coulomb collisions

Let us consider a binary collision of two particles (electron-electron, ion-ion, electron-electron). It can be considered as a scattering of a particle with a reduced mass $\mu = m_1 m_2/(m_1+m_2)$ at the potential $U = q_1 q_2/r^2$. The reduced mass is $m_e/2 \sim m_e$, $m_i/2 \sim m_i$, and m_e for $e-e$, $i-i$, and $e-i$ collisions, respectively, while $|q_1 q_2| = e^2$. The scattered particle comes from the infinity with the velocity $v = |\boldsymbol{v}_1 - \boldsymbol{v}_2|$. Let b be the impact parameter (the smallest distance from the center in the absence of interaction). If the scattering angle is small one can estimate it as $\theta \sim \Delta p_\perp/mv$, where

$$\Delta p_\perp \approx F\Delta \approx \frac{q_1 q_2}{b^2}(b/v) \sim \frac{q_1 q_2}{bv} \tag{1.15}$$

so that

$$\theta \sim q_1 q_2 mbv^2 \tag{1.16}$$

The differential cross-section of the scattering is

$$\frac{d\sigma}{d\theta} = 2\pi b db, \quad b = b(\theta) \tag{1.17}$$

The approximation of the small angle scattering is valid when $|\theta| \lesssim 1$ (or, equivalently, $|q_1 q_2|/b \lesssim mv^2$), so that

$$b > b_{min} = \frac{|q_1 q_1|}{mv^2} \tag{1.18}$$

On the other hand, Debye screening limits the impact parameter from above so that $b_{max} = r_D$. Since $b_{max}/b_{min} \gg 1$ (compare with the plasma parameter !), the ratio of the total-cross section of the scattering to small angles $\sigma(|\theta| \lesssim 1)$ to the total-cross section of the scattering to large angles $\sigma(|\theta| \gtrsim 1)$ is

$$\frac{\sigma(|\theta| \lesssim 1)}{\sigma(|\theta| \gtrsim 1)} = \left(\frac{b_{max}}{b_{min}}\right)^2 \gg 1 \tag{1.19}$$

Thus, almost all collisions on a single center are small angle collisions.

A charged particle in a plasma is being multiply scattered by a number of randomly distributed centers. The total deflection is $\Delta\theta = \sum_i \Delta\theta_i$, where $\Delta\theta_i$ is the deflection during the collision at i-th center. Since the centers are distributed randomly, the average $< \Delta\theta >= 0$. However, the variance

$$< \Delta\theta >^2 = \sum_i (\Delta\theta_i)^2 \neq 0 \tag{1.20}$$

This average can be calculated by multiplying

$$(\Delta\theta_i)^2 = \left(\frac{e^2}{mbv^2}\right) \tag{1.21}$$

by the number of collisions $2\pi nLbdb$ for each b and integrating over b:

$$(\Delta\theta)^2_M \equiv < \Delta\theta >^2 = \int_{b_{min}}^{b_{max}} 2\pi nLbdb \left(\frac{e^2}{mbv^2}\right)^2 \tag{1.22}$$

$$(\Delta\theta)^2_M = 2\pi nL \left(\frac{e^2}{mv^2}\right)^2 \Lambda \tag{1.23}$$

$$\Lambda = \ln \frac{b_{max}}{b_{min}} \gg 1 \tag{1.24}$$

Putting $(\Delta\theta)^2_M \sim 1$ one finds the distance on which the angular deflection is large:

$$L_M(|\Delta\theta| \gtrsim 1) = [2\pi n \left(\frac{e^2}{mv^2}\right)^2 \Lambda]^{-1} \tag{1.25}$$

The cross section for the multiple scattering to a large angle is

$$\sigma_M(\theta \gtrsim 1) = 1/nL_M(|\Delta\theta| \gtrsim 1)$$

therefore

$$\frac{\sigma_M(\theta \gtrsim 1)}{\sigma(\theta \gtrsim 1)} \sim \Lambda \gg 1 \tag{1.26}$$

which means that Coulomb scattering is determined by multiple small angle deflections.

The collision frequency is $\nu = nv\sigma$. Substituting $v \sim (T_e/m_e)^{1/2}$ for $e - e$ and $e - i$ collisions, and $v \sim (T_i/m_i)^{1/2}$ for $i - i$ collisions, one gets

$$\nu_{ee} \sim n \left(\frac{e^2}{T_e}\right)^2 (T_e/m_e)\Lambda_e \tag{1.27}$$

$$\nu_{ei} \sim n \left(\frac{e^2}{T_e}\right)^2 (T_e/m_e)\Lambda_e \sim \nu_{ee} \tag{1.28}$$

$$\nu_{ii} \sim n \left(\frac{e^2}{T_i}\right)^2 (T_i/m_i)\Lambda_i \sim \nu_{ee}(m_e/m_i)^{1/2} \tag{1.29}$$

In each collision of identical particles (ee or ii) the energy exchange is of the order of the particle energy. In each ei collision the energy exchange is (m_e/m_i) of the particle energy. Therefore, the time scale of the electron thermalization τ_{ee}, the ion thermalization τ_{ii} and electron-ion temperature equilibration τei is

$$\tau_{ee} : \tau_{ii} : \tau_{ei} = 1 : (m_i/m_e)^{1/2} : (m_i/m_e) \tag{1.30}$$

The electron thermalization is the fastest, the temperature equilibration is the slowest.

1.7 Summary

- Plasma is a gas of ionized particles.
- Debye length (single species): $r_D = (T/4\pi n q^2)^{1/2}$.
- Debye screening of a test particle: $\phi = Q \exp(-r/r_D)/r$.
- Plasma parameter: $g = 1/nr_D^3 \ll 1$.
- Plasma frequency (single species): $\omega_p = (4\pi n q^2/m)^{1/2}$.

1.8 Problems

PROBLEM 1.1. Calculate the Debye length for a multi-species plasma: n_s, q_s, T_s. The plasma is quasi-neutral: $\sum_s n_s q_s = 0$.

PROBLEM 1.2. Calculate the plasma frequency for a multi-species plasma: n_s, q_s, m_s. The plasma is quasi-neutral: $\sum_s n_s q_s = 0$.

PROBLEM 1.3. Calculate r_D, g and ω_p for the plasmas in Table A.

PROBLEM 1.4. A parallel plate capacitor charged to $\pm\sigma$ is immersed into an electron plasma (immobile ions). What is the potential distribution inside the capacitor ? What is its capacity ?

Chapter 2

Plasma description

In this chapter we learn about possible methods of plasma description, and derive the powerful but limited MHD.

2.1 Hierarchy of descriptions

In order to deal with plasma we have to choose some method of description. The most straightforward and most complete would seem to use the motion equations for all particles together with the Maxwell equations for the electromagnetic fields. However, it is impossible as well as unreasonable for a many-particle system with a collective behavior. A less precise but much more efficient description would be to describe all particles of the same species as a fluid in the phase space $(\boldsymbol{r}, \boldsymbol{p})$. This would correspond to the assumption that on average behavior of each particle is the same and independent of other particles, following only the prescriptions of the self-consistent fields. In this approach we forget about the possible influence of the deviations of the fields from the self-consistent values (fluctuations) and direct (albeit weak) dependence of a particle on its neighbors (correlations). This is the so-called kinetic description.

The further step toward even greater simplification of the plasma description would be to average over momenta for each species, so that only average values remain. In this case each species s is described by the local density n_s, local fluid velocity $\boldsymbol{v}_s$, local temperature T_s or pressure p_s. This is the so-called multi-fluid description.

Finally, we can even forget that there are different species and describe the plasma as one fluid with the mass density ρ, velocity $\boldsymbol{V}$, and pressure p. It is clear that electromagnetic field should be added in some way. The rest of the chapter devoted to the description of plasma as a single fluid. The description is known as magneto-hydrodynamics (MHD) for the reasons which become clear later.

2.2 Fluid description

In order to describe a fluid we choose a physically infinitesimal volume dV surrounding the point $\boldsymbol{r}$ in the moment t. The physically infinitesimal volume should be large enough to contain a large number or particles, so that statistical averaging is possible. On the other hand, it should be small enough to not make the averaging too coarse. Without coming into details we shall assume that qualitative meaning of this "infinitesimal" volume is sufficiently clear and we can make such choice.

The fluid mass ρ density is simply the sum of the masses of all particles inside this volume divided by the volume itself, $\rho = \sum m_i/dV$. Since the result may be different for volumes chosen in different places or at different times, the density can, in general, depend on $\boldsymbol{r}$ and t. The hydrodynamical velocity of this infinitesimal volume is simply the velocity of its center of mass: $\boldsymbol{V} = \sum m_i \boldsymbol{v}_i/\rho dV$. Again, $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{r}, t)$. Pressure is produced by the random thermal motion of particles (relative to the center-of-mass) in the infinitesimal volume. In order to avoid unnecessary complications we shall assume that the pressure is isotropic, that is, described by a single scalar function $p(\boldsymbol{r}, t)$. In what follows we shall consider plasma as an ideal gas, that is, $p = nT$, where $n(\boldsymbol{r}, t)$ is the concentration and $T(\boldsymbol{r}, t)$ is the temperature. Thus, we have four *fields*: $\rho(\boldsymbol{r}, t)$, $\boldsymbol{V}(\boldsymbol{r}, t)$, $p(\boldsymbol{r}, t)$, and $T(\boldsymbol{r}, t)$, for which we have to find the appropriate evolution equations, connecting the spatial and temporal variations. For brevity we do not write the dependence $(\boldsymbol{r}, t)$ in what follows.

2.3 Mass conservation - continuity equation

We start with the derivation of the continuity equation which is nothing but the mass conservation. Let us consider some volume. The total mass inside the volume is

$$M = \int_V \rho dV \tag{2.1}$$

This mass can change only due to the flow of particles into and out of the volume. If we consider a small surface element, $\boldsymbol{dS} = \hat{\boldsymbol{n}} dS$, then the mass flow across this surface during time dt will be $dM = \rho \boldsymbol{V} dt \cdot \boldsymbol{dS}$. The total flow across the surface S enclosing the volume V from inside to outside would be

$$dJ = \oint_S \rho \boldsymbol{V} \cdot \boldsymbol{dS} dt = \int_V \mathsf{div}(\rho \boldsymbol{V}) dV dt \tag{2.2}$$

Since the flow outward results in the mass decrease, we write

$$\frac{d}{dt} \int_V \rho dV = -\int_V \mathsf{div}(\rho \boldsymbol{V}) dV \Rightarrow \tag{2.3}$$

$$\int_V \left[\partial_t \rho + \mathsf{div}(\rho \boldsymbol{V})\right] dV = 0 \Rightarrow \tag{2.4}$$

$$\partial_t \rho + \mathsf{div}(\rho \boldsymbol{V}) = 0. \tag{2.5}$$

The last relation follows from the fact that the previous should be valid for any arbitrary (including infinitesimal) volume at any time. Equation (2.5) is the continuity equation.

2.4 Momentum conservation - motion (Euler) equation

The single particle motion equation is nothing but the equation for the change of its momentum. We shall derive the equation of motion for the fluid considering the change of the momentum of the fluid in some volume V. The total momentum at any time would be

$$\boldsymbol{P} = \int_V \rho \boldsymbol{V} dV, \quad P_i = \int_V \rho U_i dV \tag{2.6}$$

The momentum changes due to the flow of the fluid across the boundary and due to the forces acting from the other fluid at the boundary. Let us start with the momentum flow. The fluid volume which flows across the surface $\boldsymbol{dS}$ during time dt is $\boldsymbol{V}dt\cdot\boldsymbol{dS}$. This flowing volume takes with it the momentum $d\boldsymbol{P} = (\rho\boldsymbol{V})(\boldsymbol{V}dt\cdot\boldsymbol{dS})$. Thus, the total flow of the momentum outward is

$$d\boldsymbol{P} = \oint_S (\rho\boldsymbol{V})(\boldsymbol{V}\cdot\boldsymbol{dS})dt \tag{2.7}$$

The total force which acts on the boundaries of the volume from the outside fluid is

$$\boldsymbol{F} = -\oint_S p\boldsymbol{dS} \tag{2.8}$$

Combining (2.6)-(2.8) we get

$$\frac{d}{dt}\int_V \rho\boldsymbol{V}dV = -\oint_S (\rho\boldsymbol{V})(\boldsymbol{V}\cdot\boldsymbol{dS}) - \oint_S p\boldsymbol{dS} \tag{2.9}$$

Further derivation is simpler if we write (2.9) in the component (index) representation:

$$\int_V \partial_t(\rho V_i)dV = -\oint_S (\rho V_i V_j)dS_j - \oint_S p\delta_{ij}dS_j \tag{2.10}$$

and use the vector analysis theorem:

$$\oint_S A_{ij}dS_j = \int_V \partial_j A_{ij}dV \tag{2.11}$$

Now we get the motion equation in the following form:

$$\partial_t(\rho V_i) + \partial_j(\rho V_i V_j) = -\partial_j p \tag{2.12}$$

If we recall that the continuity equation can be written as

$$\partial_t\rho + \partial_j(\rho V_j) = 0 \tag{2.13}$$

we can rewrite (2.12) in the following widely used form:

$$\rho\left(\partial_t\boldsymbol{V} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right) = -\,\mathsf{grad}\,p \tag{2.14}$$

One has to be cautious with the form of the equation since $(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}$ is not a good vector form and cannot be easily written in curvilinear coordinates. Instead one has to use the proper vector representation

$$(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} = \mathsf{grad}\left(\frac{V^2}{2}\right) - \boldsymbol{V}\times\mathsf{rot}\,\boldsymbol{V} \tag{2.15}$$

It is worth noting that the force $-\,\mathsf{grad}\,p$ the *volume* force, that is, the forth per unit volume. If other volume forces exist we should simply add them to the right hand side of (2.14).

2.5 State equation

We have derived 4 equations (one for the scalar and three for the vector equation) for 5 variables: ρ_m, three components of $\boldsymbol{V}$, and p. Therefore, we need another equation for the pressure p. Either we have to derive it from the first principles, as we did for the continuity equation and the motion equation, or to use some sort of approximate closure. For this course we just assume that the pressure is a function of density, $p = p(\rho)$. In most cases a polytropic dependence will be assumed $p = (\rho/\rho_0)^\gamma$.

2.6 Energy conservation

Let us denote the internal energy with $\mathcal{E}$. Let $\epsilon = d\mathcal{E}/dm$ be the internal energy per unit mass.The energy of the plasma in the volume δV would be $\rho\left(\frac{V^2}{2}+\epsilon\right)\delta V$ and the total plasma energy in the volume V would be

$$\mathcal{E} = \int_V \rho\left(\frac{V^2}{2}+\epsilon\right)dV \tag{2.16}$$

This energy changes because of the flow across the boundary and the work of the volume forces and the ambient pressure:

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= \int_V \partial_t\left[\rho\left(\frac{U^2}{2}+\epsilon\right)\right]dV \\ &= -\int_S \rho\left(\frac{V^2}{2}+\epsilon\right)\boldsymbol{V}\cdot d\boldsymbol{S} \\ &+ \int_V \boldsymbol{f}\cdot\boldsymbol{V}dV - \int_S pd\boldsymbol{S}\cdot\boldsymbol{V}\end{aligned} \tag{2.17}$$

In the ideal MHD

$$\boldsymbol{f}\cdot\boldsymbol{V} = \frac{1}{c}(\boldsymbol{j}\times\boldsymbol{B})\cdot\boldsymbol{V} = \boldsymbol{E}\cdot\boldsymbol{j} \tag{2.18}$$

For the electromagnetic energy one has

$$\partial_t\left(\frac{B^2}{8\pi}\right) + \mathsf{div}\,\frac{c\boldsymbol{E}\times\boldsymbol{B}}{4\pi} = -\boldsymbol{E}\cdot\boldsymbol{j} \tag{2.19}$$

Combining the plasma energy and electromagnetic energy one has

$$\partial_t \left[\rho \left(\frac{V^2}{2} + \epsilon \right) + \frac{B^2}{8\pi} \right] + \mathsf{div} \left\{ \left[\rho \left(\frac{V^2}{2} + w \right) + \frac{B^2}{4\pi} \right] \boldsymbol{V} \right\} = 0 \tag{2.20}$$

where $w = \epsilon + p/\rho$.

We assume that entropy is constant, then

$$d\mathcal{E} = TdS - pdV \tag{2.21}$$

Defining

$$\epsilon = \frac{d\mathcal{E}}{dm}, \quad s = \frac{dS}{dm}, \quad \frac{1}{\rho} = \frac{dV}{dm} \tag{2.22}$$

one gets

$$d\epsilon = Tds + \frac{p}{\rho^2} d\rho \tag{2.23}$$

For isentropic polytropic motions one has

$$\epsilon = \frac{p}{(\gamma - 1)\rho}, \quad w = \frac{\gamma p}{(\gamma - 1)\rho} \tag{2.24}$$

2.7 MHD

So far we have been treating a single fluid, without any relation to plasma. What makes the fluid plasma is its ability to carry currents. If the current density in the plasma is $\boldsymbol{j}$ then it experiences the Ampere force $(1/c)\boldsymbol{j} \times \boldsymbol{B}$, once the magnetic field $\boldsymbol{B}$ is present. In principle, electric volume force $\rho_q \boldsymbol{E}$ may be also present. However, in the non-relativistic MHD approximation the plasma is quasi-neutral and this term is absent, and for the rest of the chapter we do not write the index m for ρ - it is always the mass density. (If you ever wish to learn *relativistic* MHD do not forget the electric force.) Thus, the motion equation takes the following form:

$$\rho \left(\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \boldsymbol{V} \right) = - \mathsf{grad}\, p + \frac{1}{c} \boldsymbol{j} \times \boldsymbol{B} \tag{2.25}$$

However, now we have two more vector variables: $\boldsymbol{j}$ and $\boldsymbol{B}$. It is time to add the Maxwell equations:

$$\mathsf{div}\, \boldsymbol{B} = 0, \tag{2.26}$$

$$c\, \mathsf{rot}\, \boldsymbol{B} = 4\pi \boldsymbol{j} + \partial_t \boldsymbol{E}, \tag{2.27}$$

$$c\, \mathsf{rot}\, \boldsymbol{E} = -\partial_t \boldsymbol{B}. \tag{2.28}$$

We do not need the $\mathsf{div}\, \boldsymbol{E} = 4\pi \rho_q$ equation since quasi-neutrality is assumed and this equation does not add to the dynamic evolution equations, but rather allows to check the assumption in the end of calculations. Eq. (2.26) is a constraint, not an evolution equation since it does not include time derivative. It is also redundant since (2.28) shows that once $(\partial\, \mathsf{div}\, \boldsymbol{B}/\partial t) = 0$, and once (2.26) is satisfied initially it will be satisfied forever.

It can be shown (we shall see that later in the course) that non-relativistic MHD is the limit of slow motions and large scale spatial derivatives, so that the displacement current is always negligible, and (2.27) becomes a relation between the magnetic field and current density

$$\boldsymbol{j} = \frac{c}{4\pi} \,\mathsf{rot}\, \boldsymbol{B}. \tag{2.29}$$

The only evolution equation which remains is the induction equation (2.28). However, it includes now the new variable $\boldsymbol{E}$ which does not seem to be otherwise related to any other variable. Ohm's law comes to help. The local Ohm's law for a immobile conductor is written as $\boldsymbol{j} = \sigma \boldsymbol{E}$. Plasma is a moving conductor and the Ohm's law should be written in the plasma rest frame, $\boldsymbol{j}' = \sigma \boldsymbol{E}'$. For non-relativistic flows the rest frame electric field $\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}/c$, while $\boldsymbol{j}' = \boldsymbol{j}$ because of the quasi-neutrality condition. Thus, the Ohm's law should be written in our case as

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}/c). \tag{2.30}$$

This relation is used to express the electric field in terms of the magnetic field:

$$\boldsymbol{E} = -\frac{1}{c}\boldsymbol{V} \times \boldsymbol{B} + \frac{c}{4\pi\sigma} \,\mathsf{rot}\, \boldsymbol{B}, \tag{2.31}$$

and substitute this in (2.28):

$$\partial_t \boldsymbol{B} = \mathsf{rot}(\boldsymbol{V} \times \boldsymbol{B}) + \frac{c^2}{4\pi\sigma}\Delta \boldsymbol{B}, \tag{2.32}$$

thus getting an equation containing only $\boldsymbol{B}$ and $\boldsymbol{V}$.

Now, substituting (2.29) into (2.25) we get the equation of motion free of the current:

$$\rho\left(\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}\right) = -\,\mathsf{grad}\, p + \frac{1}{4\pi} \,\mathsf{rot}\, \boldsymbol{B} \times \boldsymbol{B}. \tag{2.33}$$

2.8 Order-of-magnitude estimates

Let L be the typical inhomogeneity length, which means that when we move by $\Delta x, y, z \sim L$ the variable under consideration, say $\boldsymbol{B}$ changes by $\Delta B \sim B$. Now substitute $(\partial \boldsymbol{B}/\partial x) \sim \Delta B/\Delta x \sim B/L$, that is $\nabla \sim 1/L$. Similarly, if T is the typical variation time, we have $(\partial/\partial t) \sim 1/T$. Typical velocity then is estimated as $U \sim L/T$. Using these definitions we can estimate from the induction equation $E \sim (U/c)B$. Respectively, the ratio of the displacement current to $\mathsf{rot}\, \boldsymbol{B}$ term will be

$$|\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}|/|\,\mathsf{rot}\, \boldsymbol{B}| \sim \frac{LE}{cTB} \sim \left(\frac{U}{c}\right)^2$$

and is very small for nonrelativistic velocities. This is the reason, why it is usually neglected.

For the charge density we have $\rho_q = \mathsf{div}\, \boldsymbol{E}/4\pi \sim E/4\pi L$. Thus, the ratio of the electric and magnetic forces

$$\frac{|\rho_q \boldsymbol{E}|}{|(1/c)\boldsymbol{j} \times \boldsymbol{B}|} \sim \frac{1}{4\pi}\left(\frac{U}{c}\right)^2$$

and is also negligible.

2.9 Summary

Let us write down again the complete set of the MHD equations:

$$\partial_t \rho + \mathsf{div}(\rho \boldsymbol{V}) = 0, \tag{2.34}$$

$$\rho \frac{d}{dt}\boldsymbol{V} = -\,\mathsf{grad}\, p + \frac{1}{4\pi}\,\mathsf{rot}\,\boldsymbol{B} \times \boldsymbol{B}, \tag{2.35}$$

$$\partial_t \boldsymbol{B} = \mathsf{rot}(\boldsymbol{V} \times \boldsymbol{B}) + \frac{c^2}{4\pi\sigma}\Delta \boldsymbol{B}, \tag{2.36}$$

where we introduced the substantial derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}). \tag{2.37}$$

The MHD set should be completed with the state equation $p = p(\rho)$ and is usually completed with the Ohm's law $\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}/c = \boldsymbol{j}/\sigma$. When $\sigma \to \infty$ the MHD is *ideal* MHD.

2.10 Problems

PROBLEM 2.1. Complete the MHD equations for the case when there is gravity.

PROBLEM 2.2. For $p \propto \rho^\gamma$ and no entropy change show that the internal energy per unit volume $u = p/(\gamma - 1)$.

PROBLEM 2.3. Derive the energy conservation:

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2 + u + \frac{B^2}{8\pi}\right) + \mathsf{div}\left((\tfrac{1}{2}\rho V^2 + u + p)\boldsymbol{V} + \frac{1}{4\pi}\boldsymbol{B} \times (\boldsymbol{V} \times \boldsymbol{B})\right) = 0 \tag{2.38}$$

PROBLEM 2.4. Let a plasma penetrate a neutral fluid. Discuss the form of the frictional force between the two fluids in the equation of motion for the plasma.

Chapter 3

MHD equilibria and waves

In this chapter we become familiar with the coordinated behavior of plasma and magnetic field, and discover the most important features of the plasma - waves.

3.1 Magnetic field diffusion and dragging

We start our study of MHD applications with the analysis of (2.36). Let us consider first the case where the plasma is not moving at all, that is, $\boldsymbol{V} = 0$. Fot simplicity let $\boldsymbol{B} = B(x,t)\hat{\boldsymbol{z}}$, so that one gets

$$\frac{\partial}{\partial t}B = \frac{c^2}{4\pi\sigma}\frac{\partial^2}{\partial x^2}B. \tag{3.1}$$

Let us represent the magnetic field using Fourier-transform:

$$B(x,t) = \int_{-\infty}^{\infty} \tilde{B}(k,t)\exp(ikx)dk, \tag{3.2}$$

then one has

$$B(\dot{\tilde{k}},t) = -\frac{k^2c^2}{4\pi\sigma}B(\tilde{k},t) \tag{3.3}$$

with the solution

$$B(\tilde{k},t) = B(\tilde{k},0)\exp(-k^2c^2t/4\pi\sigma). \tag{3.4}$$

The *wavenumber* k is the measure of spatial inhomogeneity: the larger k the smaller is the inhomogeneity scale. Eq. (3.4) shows that the inhomogeneous magnetic field disappears with time, and the rate of disappearance higher for the components with smaller scales of inhomogeneity. This phenomenon is known as the magnetic field diffusion and is responsible for the graduate dissipation of the magnetic fields in stars.

Let us no consider the opposite case: $\sigma \to \infty$ and $\boldsymbol{V} \neq 0$. Let us choose a closed path (contour) L moving with the plasma and calculate the change of the magnetic flux across the surface S enclosed by this contour, $\Phi = \int_S \boldsymbol{B}\cdot\boldsymbol{dS}$. The flux changes due to the local change of the magnetic field and due to the change of the contour moving with the plasma. The total change during the time dt is

$$d\Phi = \int_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot\boldsymbol{dS}dt + \oint_L \boldsymbol{B}\cdot(\boldsymbol{V}dt\times\boldsymbol{dL}) \tag{3.5}$$

$$= \left(\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{dS} - \oint_L (\boldsymbol{V} \times \boldsymbol{B}) \cdot \boldsymbol{dL} \right) dt \tag{3.6}$$

$$= \int_S \left(\frac{\partial \boldsymbol{B}}{\partial t} - \mathsf{rot}(\boldsymbol{V} \times \boldsymbol{B}) \right) \cdot \boldsymbol{dS} dt = 0 \tag{3.7}$$

that is, the magnetic flux across the contour moving with the plasma, does not change. This is often referred to as the magnetic field frozen in plasma: magnetic field lines are dragged by plasma. For the rest of the course we will be dealing with the ideal MHD only, if not stated explicitly otherwise.

3.2 Equilibrium conditions

Plasma is said to be in the equilibrium if $\boldsymbol{V} = 0$ and none of the variables depend on time. The only equation which has to be satisfied is

$$\mathsf{grad}\, p = \frac{1}{c} \boldsymbol{j} \times \boldsymbol{B} = \frac{1}{4\pi} \mathsf{rot}\, \boldsymbol{B} \times \boldsymbol{B}. \tag{3.8}$$

One can immediately see that in the equilibrium $\mathsf{grad}\, p \perp \boldsymbol{B}$ and $\mathsf{grad}\, p \perp \boldsymbol{j}$, that is, the current lines and the magnetic field lines all lie on the constant pressure surfaces. In the special case $\boldsymbol{j} \parallel \boldsymbol{B}$ no pressure forces are necessary to maintain the equilibrium, the configuration is called force-free.

The right hand side of (3.8) is often casted the in the following form:

$$\frac{1}{4\pi} \mathsf{rot}\, \boldsymbol{B} \times \boldsymbol{B} = -\,\mathsf{grad}\, \frac{B^2}{8\pi} + \frac{1}{4\pi} (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B}, \tag{3.9}$$

where the first term represents the *magnetic pressure*, while the last one is the *magnetic tension*.

In order to understand better the physical sense of the two terms let start with considering the magnetic field of the form $\boldsymbol{B} = (B_x, B_y, 0)$ and assume that everything depends on x only. Then (3.8) with (3.9) read

$$\frac{d}{dx} \left(p + \frac{B^2}{8\pi} \right) = \frac{1}{4\pi} B_x \frac{d}{dx} B_x, \tag{3.10}$$

$$0 = B_x \frac{d}{dx} B_y \tag{3.11}$$

Since $\mathsf{div}\, \boldsymbol{B} = \frac{dB_x}{dx} = 0$ we have only two options: a) $\boldsymbol{B} = \text{const}$ and $p = \text{const}$ (not interesting), and b) $B_x = 0$, $B_y = B_y(x)$, and

$$p + \frac{B_y^2}{8\pi} = \text{const}. \tag{3.12}$$

Thus, in this case the direction of the magnetic field does not change, and mechanical equilibrium requires that the total (gas+magnetic) pressure be constant throughout.

3.3 MHD waves

Waves are the heart of plasma physics. There is nothing which plays a more important role in plasma life than waves, small or large amplitude ones. The rest of this chapter is devoted to the description of the wave properties of plasmas within the MHD approximation.

As in other media, waves are small perturbations which propagate in the medium. Thus, a medium which is perturbed in some place initially would be perturbed in other place later. In order to study waves we have to learn to deal with small perturbations near some equilibrium. We outline here the general procedure of the wave equations derivation, the procedure we shall closely follow later in our studies of waves in more sophisticated descriptions.

Step 1. Equilibrium. We start with the equilibrium state, where nothing depends on time and there no flows. In our course we shall study only waves in homogeneous plasmas, that is, we assume that the background (equilibrium) plasma parameters do not depend on coordinates either. In the MHD case that means $\rho = \rho_0 = \text{const}$, $\boldsymbol{V} = 0$, $p = p_0 = \text{const}$, and $\boldsymbol{B} = \boldsymbol{B}_0 = \text{const}$.

Step 2. Small perturbations. We assume that all variables are slightly perturbed: $\rho = \rho_0 + \epsilon\rho_1$, $\boldsymbol{V} = \epsilon\boldsymbol{V}_1$, $p = p_0 + \epsilon p_1$, and $\boldsymbol{B} = \boldsymbol{B}_0 + \epsilon\boldsymbol{B}_1$, where $\epsilon \ll 1$ is a formal small parameter which will allow us to collect terms which are of the same order of magnitude (see below). We have to substitute the perturbed quantities into the MHD equations (2.34)-(2.36):

$$\frac{\partial}{\partial t}(\epsilon\rho_1) + \mathsf{div}(\epsilon\rho_0\boldsymbol{V}_1 + \epsilon^2\rho_1\boldsymbol{V}_1) = 0, \tag{3.13}$$

$$\begin{aligned}&(\rho_0 + \epsilon\rho_1)\frac{\partial}{\partial t}(\epsilon\boldsymbol{V}_1) + (\epsilon\boldsymbol{V}_1\cdot\boldsymbol{\nabla})(\epsilon\boldsymbol{V}_1)\\ &= -\,\mathsf{grad}(\epsilon p_1) + \frac{1}{4\pi}\,\mathsf{rot}(\epsilon\boldsymbol{B}_1)\times(\boldsymbol{B}_0 + \epsilon\boldsymbol{B}_1),\end{aligned} \tag{3.14}$$

$$\frac{\partial}{\partial t}(\epsilon\boldsymbol{B}_1) = \mathsf{rot}(\epsilon\boldsymbol{V}_1\times\boldsymbol{B}_0 + \epsilon^2\boldsymbol{V}_1\times\boldsymbol{B}_1), \tag{3.15}$$

where we have taken into account that all derivatives of the unperturbed variables (index 0) vanish.

Step 3. Linearization. This is one of the most important steps, were we neglect all terms of the order ϵ^2 and higher and retain only the *linear* terms $\propto \epsilon$, to get

$$\frac{\partial}{\partial t}\rho_1 + \rho_0\,\mathsf{div}\,\boldsymbol{V}_1 = 0, \tag{3.16}$$

$$\rho_0\frac{\partial}{\partial t}\boldsymbol{V}_1 = -\,\mathsf{grad}\,p_1 + \frac{1}{4\pi}\,\mathsf{rot}\,\boldsymbol{B}_1\times\boldsymbol{B}_0, \tag{3.17}$$

$$\frac{\partial}{\partial t}\boldsymbol{B}_1 = \mathsf{rot}(\boldsymbol{V}_1\times\boldsymbol{B}_0). \tag{3.18}$$

We have to find a relation between p_1 and ρ_1. It simply follows from the Taylor expansion (first term):

$$p_1 = \left(\frac{dp}{d\rho}\right)_{\rho=\rho_0} \rho_1 \equiv v_s^2 \rho_1, \tag{3.19}$$

where the physical meaning of the quantity v_s^2 will become clear later.

Step 4. Fourier transform. The obtained equations are linear differential equations with constant coefficients, and the usual way of solving these equations is to assume for all variables the same dependence $\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$, that is, $\rho_1 = \tilde{\rho}_1 \exp(i\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$, etc. Here $\boldsymbol{k}$ is the *wavevector* and ω is *frequency*. It is easy to see that one has to simply substitute $(\partial/\partial t) \to -i\omega$ and $\boldsymbol{\nabla} \to i\boldsymbol{k}$, so that

$$-i\omega\tilde{\rho}_1 + i\rho_0 \boldsymbol{k}\cdot\tilde{\boldsymbol{V}}_1 = 0, \tag{3.20}$$

$$-i\omega\rho_0\tilde{\boldsymbol{V}}_1 = -iv_s^2\boldsymbol{k}\tilde{\rho}_1 + \frac{i}{4\pi}(\boldsymbol{k}\times\tilde{\boldsymbol{B}}_1)\times\boldsymbol{B}_0, \tag{3.21}$$

$$-i\omega\tilde{\boldsymbol{B}}_1 = i\boldsymbol{k}\times(\tilde{\boldsymbol{V}}_1\times\boldsymbol{B}_0). \tag{3.22}$$

The obtained equations are a homogeneous set of 6 equations for 6 variables: the density, three components of the velocity, and two independent components of the magnetic field - third is dependent because of $\mathsf{div}\,\boldsymbol{B}_1 = 0 \Rightarrow i\boldsymbol{k}\cdot\tilde{\boldsymbol{B}}_1 = 0$.

Step 5. Dispersion relation. In order for non-trivial (nonzero) solutions to exist the determinant for this set should be equal zero. This determinant is a function of the unperturbed parameters as well as ω and $\boldsymbol{k}$. Let us assume that the determinant calculation provided us with the equation

$$D(\omega, \boldsymbol{k}) = 0. \tag{3.23}$$

This equation established a relation between the frequency and the wavevector, for which a nonzero solution can exist. This relation (and often (3.23) itself) is called a *dispersion relation.*

It is possible to write down the 6×6 determinant derived directly from (3.20)-(3.22). However, it is more instructive and physically transparent to look at the magnetic field and velocity components. Eq. (3.20) shows that density (and pressure) variations are related only to the velocity component along the wavevector,

$$\tilde{\rho}_1 = \rho_0(\boldsymbol{k}\cdot\tilde{\boldsymbol{V}}_1)/\omega. \tag{3.24}$$

Eq. (3.22) shows that the magnetic field perturbations are always perpendicular to the wavevector, $\tilde{\boldsymbol{B}}_1 \perp \boldsymbol{k}$.

The subsequent derivation is a little bit long but rather straightforward and physically transparent. It is convenient to define a new variable $\boldsymbol{E} = \boldsymbol{k}\times\tilde{\boldsymbol{B}}_1$, such that $\boldsymbol{k}\cdot\boldsymbol{E} = 0$. The equations take the following form:

$$-\omega\rho_0\tilde{\boldsymbol{V}}_1 = -\frac{v_s^2\rho_0\boldsymbol{k}}{\omega}(\boldsymbol{k}\cdot\tilde{\boldsymbol{V}}_1) + \frac{1}{4\pi}\boldsymbol{E}\times\boldsymbol{B}_0, \tag{3.25}$$

$$-\omega\boldsymbol{E} = (\boldsymbol{k}\times\tilde{\boldsymbol{V}}_1)(\boldsymbol{k}\cdot\boldsymbol{B}_0) - (\boldsymbol{k}\times\boldsymbol{B}_0)(\boldsymbol{k}\cdot\tilde{\boldsymbol{V}}_1). \tag{3.26}$$

Scalar and vector products of with $\boldsymbol{k}$ give, respectively:

$$\left(1-\frac{k^2v_s^2}{\omega^2}\right)(\boldsymbol{k}\cdot\tilde{\boldsymbol{V}}_1)=\frac{1}{4\pi\rho_0\omega}\boldsymbol{E}\cdot(\boldsymbol{k}\times\boldsymbol{B}_0), \tag{3.27}$$

$$\boldsymbol{k}\times\tilde{\boldsymbol{V}}_1=-\frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)}{4\pi\rho_0\omega}\boldsymbol{E}. \tag{3.28}$$

Substituting (3.28) into (3.26) one obtains

$$\left(1-\frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{4\pi\rho_0\omega^2}\right)\boldsymbol{E}=(\boldsymbol{k}\times\boldsymbol{B}_0)(\boldsymbol{k}\cdot\tilde{\boldsymbol{V}}_1). \tag{3.29}$$

Now the scalar and vector products of (3.29) with $\boldsymbol{k}\times\boldsymbol{B}_0$ give

$$\left(1-\frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{4\pi\rho_0\omega^2}\right)\boldsymbol{E}\times(\boldsymbol{k}\times\boldsymbol{B}_{\boldsymbol{0}})=0, \tag{3.30}$$

$$\left(1-\frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{4\pi\rho_0\omega^2}\right)\boldsymbol{E}\cdot(\boldsymbol{k}\times\boldsymbol{B}_{\boldsymbol{0}})=(\boldsymbol{k}\times\boldsymbol{B}_0)^2(\boldsymbol{k}\cdot\tilde{\boldsymbol{V}}_1). \tag{3.31}$$

Equation (3.30) means that $\boldsymbol{E}\times(\boldsymbol{k}\times\boldsymbol{B}_{\boldsymbol{0}})\neq 0$ only if

$$\omega^2=\frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{4\pi\rho_0}, \tag{3.32}$$

while (3.31) together with (3.27) give for $\boldsymbol{E}\times(\boldsymbol{k}\times\boldsymbol{B}_{\boldsymbol{0}})=0$

$$\left(1-\frac{k^2v_s^2}{\omega^2}\right)\left(1-\frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{4\pi\rho_0\omega^2}\right)=\frac{(\boldsymbol{k}\times\boldsymbol{B}_0)^2}{4\pi\rho_0\omega^2}. \tag{3.33}$$

Before we start analyzing the derived dispersion relations let us introduce some useful notation: $\widehat{(\boldsymbol{k}\boldsymbol{B}_0)}=\theta$, so that $\boldsymbol{k}\cdot\boldsymbol{B}_0=kB_0\cos\theta$, $|\boldsymbol{k}\times\boldsymbol{B}_0|=kB_0\sin\theta$. We also define the Alfven velocity as $v_A^2=B_0^2/4\pi\rho_0$. The wave phase velocity $\boldsymbol{v}_{ph}=(\omega/k)\hat{\boldsymbol{k}}$, $v_{ph}=\omega/k$. Now (3.32) takes the following form:

$$v_{ph}^2=v_I^2\equiv v_A^2\cos^2\theta, \tag{3.34}$$

where index I stands for *intermediate*. It is easy to see that for this wave $\boldsymbol{B}_1\perp\boldsymbol{B}_0$, so that the perturbation of the magnetic field magnitude $\delta B^2=2\boldsymbol{B}_0\cdot\boldsymbol{B}_1=0$, hence the magnetic pressure does not change. Similarly, $\boldsymbol{V}_1\perp\boldsymbol{k}$ and there are no perturbations of the density and plasma pressure.

The relation (3.33) gives $v_{ph}=v_F$ (for *fast*) or $v_{ph}=v_{SL}$ (for *slow*), where

$$v_{F,SL}^2=\tfrac{1}{2}\left[(v_A^2+v_s^2)\pm\sqrt{(v_A^2+v_s^2)^2-4v_A^2v_s^2\cos^2\theta}\right]. \tag{3.35}$$

The two modes are compressible, $\rho_1\neq 0$, $p_1\neq 0$, and $\boldsymbol{B}_1$ lies in the plane of $\boldsymbol{k}$ and $\boldsymbol{B}_0$. The names of the modes are related to the fact that $v_{SL}<v_I<v_F$.

It is easy to see that if there is no external magnetic field, $B_0=0$, the only possible mode is $v_{ph}^2=v_s^2$. In the ordinary gas this wave mode would be just *sound*, that is, propagating pressure perturbations, so that v_s is the *sound velocity*. In the presence of the magnetic field the magnetic pressure and the gas pressure either act in the same phase (in the fast wave) or in the opposite phases (in the slow wave).

3.4 Alfven and magnetosonic modes

We start our analysis with the intermediate mode, which is also called Alfven wave. The dispersion relation reads

$$\omega = kv_A \cos\theta = (\boldsymbol{k} \cdot \hat{\boldsymbol{b}})v_A, \tag{3.36}$$

where $\hat{\boldsymbol{b}} = \boldsymbol{B}_0/B_0$. The magnetic field perturbations $\boldsymbol{B}_1 \perp \boldsymbol{k}, \boldsymbol{B}_0$ and there are no density perturbations. The velocity perturbations

$$\tilde{\boldsymbol{V}}_1 = -v_A\tilde{\boldsymbol{B}}_1/B_0. \tag{3.37}$$

The phase velocity $\boldsymbol{v}_{ph} = (\omega/k)\hat{\boldsymbol{k}} = v_A \cos\theta\hat{\boldsymbol{k}}$, while the group velocity (the velocity with which the energy is transferred by a wave packet) is

$$\boldsymbol{v}_g = \frac{d\omega}{d\boldsymbol{k}} = v_A\hat{\boldsymbol{b}}, \tag{3.38}$$

and is directed along the magnetic field. To summarize, Alfven waves are magnetic perturbations, whose energy propagates along the magnetic field. Plasma remains incompressible in this mode. This perturbations become non-propagating (do not exist) when $\boldsymbol{k} \perp \boldsymbol{B}_0$. The last statement means also that the slow mode does not exist either for the perpendicular propagation.

The two other modes are both *magnetosonic* waves, since they combine magnetic perturbations with the density and pressure perturbations, typical for sound waves. In the case of perpendicular propagation only the fast mode exists with

$$v_F = \sqrt{v_A^2 + v_s^2}, \tag{3.39}$$

while the parallel case, $\boldsymbol{k} \parallel \boldsymbol{B}_0$ both are present with

$$v_F = \max(v_A, v_s), \qquad v_{SL} = \min(v_A, v_s). \tag{3.40}$$

In the fast wave the perturbations of the magnetic field and density are in phase, that is, increase of the magnetic field magnitude is accompanied by the density increase. In the slow mode the magnetic field increase causes the density decrease.

It is worth mentioning that the ratio v_s/v_A depends on the kinetic-to-magnetic pressure ratio. Let us assume, for simplicity, a polytropic law for the pressure: $p = p_0(\rho/\rho_0)^\gamma$, then $v_s^2 = \gamma p_0/\rho_0$. Thus,

$$\frac{v_s^2}{v_A^2} = \frac{4\pi\gamma p_0}{B_0^2} = \frac{\gamma p_0}{2p_B}. \tag{3.41}$$

It is widely accepted to denote $\beta = p_0/p_B = 8\pi p_0/B_0^2$.

3.5 Wave energy

For simplicity we consider only the incompressible Alfven mode here. The general solution for the magnetic field can be written as

$$\boldsymbol{B}_1 = \int \tilde{\boldsymbol{B}}_1 \exp[i\boldsymbol{k} \cdot \boldsymbol{r} - \omega(\boldsymbol{k})t]d\boldsymbol{k}. \tag{3.42}$$

The corresponding velocity will be written as follows

$$\boldsymbol{V}_1 = -(v_A/B_0)\int \tilde{\boldsymbol{B}}_1 \exp[i\boldsymbol{k}\cdot\boldsymbol{r} - \omega(\boldsymbol{k})t]d\boldsymbol{k}. \tag{3.43}$$

The energy density is $u = \rho\boldsymbol{V}^2/2 + \boldsymbol{B}^2/8\pi$, which gives

$$\delta u = u - u_0 = \rho_0\boldsymbol{V}_1^2/2 + \boldsymbol{B}_0\cdot\boldsymbol{B}_1/4\pi + \boldsymbol{B}_1^2/8\pi,$$

while the total energy is

$$U = \int(\rho_0\boldsymbol{V}_1^2/2 + \boldsymbol{B}_0\cdot\boldsymbol{B}_1/4\pi + \boldsymbol{B}_1^2/8\pi)dV.$$

The second term vanishes because of the oscillations of the integrand. The other two terms become

$$U = \int(\rho_0 v_A^2/2B_0^2 + 1/8\pi)|\tilde{\boldsymbol{B}}_1|^2 d\boldsymbol{k}$$

which allows to define the quantity

$$U_{\boldsymbol{k}} = |\tilde{\boldsymbol{B}}_1|^2/4\pi \tag{3.44}$$

as the Alfven wave energy. Thus, in the Alfven wave the energy density of plasma motions equals the energy density of the magnetic field.

3.6 Summary

3.7 Problems

PROBLEM 3.1. An infinitely long cylinder of plasma, with the radius R, carries current with the uniform current density $\boldsymbol{J} = J\hat{\boldsymbol{z}}$ along the axis. Find the pressure distribution required for equilibrium.

PROBLEM 3.2. Magnetic field is given as $\boldsymbol{B} = B_0\tanh(x/d)\hat{\boldsymbol{y}}$. Find the current and density distribution if $p = C\rho^\gamma$.

PROBLEM 3.3. A plasma is embedded in a homogeneous gravity field $\boldsymbol{g}$. How the equilibrium conditions are changed.

PROBLEM 3.4. A plasma with the conductivity σ is embedded in the magnetic field of the kind $\boldsymbol{B} = \hat{\boldsymbol{y}}B_0\tanh(x/D)$ at $t = 0$. Find the magnetic field evolution if there is no plasma flows.

PROBLEM 3.5. Derive the phase and group velocities for both magnetosonic modes.

PROBLEM 3.6. Express the condition $|(1/c)(\partial \boldsymbol{E}/\partial t)| \ll |\,\mathsf{rot}\,\boldsymbol{B}|$ with the use of the Alfven velocity.

PROBLEM 3.7. Derive the dispersion relations for $v_s = v_A$.

PROBLEM 3.8. Determine the magnetic field of a cylindrically symmetric configuration as a function of distance from the axis: $\boldsymbol{B}(r) = B_z(r)\hat{\boldsymbol{z}} + B_\varphi(r)\hat{\boldsymbol{\varphi}}$. Assume a force-free field configuration of $\mathsf{rot}\,\boldsymbol{B} = \alpha \boldsymbol{B}$, where $\alpha = \text{const}$.

PROBLEM 3.9. Derive dispersion relations for MHD waves in the case when the resistivity $\eta = 1/\sigma \neq 0$.

PROBLEM 3.10. Calculate the ratio of plasma pressure perturbation to the magnetic pressure perturbation for magnetosonic waves ?

PROBLEM 3.11. Find the electric field vector for MHD waves.

Chapter 4

MHD discontinuities

MHD describes not only small amplitude waves but also large amplitude structures. In this chapter we shall study discontinuities.

4.1 Stationary structures

A wave (or structure) is said to be stationary if there is an inertial frame where nothing depends on time, $(\partial/\partial t) = 0$. Moreover, in most cases it is assumed that all variables depend on one coordinate only. Let us choose coordinates so that everything depends only on x. Then $\boldsymbol{\nabla} = \hat{\boldsymbol{x}}\frac{d}{dx}$ and the MHD equations can be written as follows:

$$\frac{d}{dx}(\rho V_x) = 0, \tag{4.1}$$

$$\rho V_x \frac{d}{dx}\left(\rho V_x^2 + p + \frac{B^2}{8\pi}\right) = 0, \tag{4.2}$$

$$\frac{d}{dx}\left(\rho V_x \boldsymbol{V}_\perp - \frac{B_x}{4\pi}\boldsymbol{B}_\perp\right) = 0, \tag{4.3}$$

$$\frac{d}{dx}\left(B_x \boldsymbol{V}_\perp - V_x \boldsymbol{B}_\perp\right) = 0, \tag{4.4}$$

$$\frac{d}{dx}\left(\rho\left(\frac{V^2}{2} + w\right)V_x + \frac{B^2}{4\pi}V_x\right) = 0 \tag{4.5}$$

where $\perp$ stands for y and z components, and B_x = const because of $\mathsf{div}\,\boldsymbol{B} = (\partial B_x/\partial x) = 0$.

Eqs. (4.1)-(4.4) can be immediately integrated to give

$$\rho V_x = J = \text{const}, \tag{4.6}$$

$$\rho V_x^2 + p + \frac{B^2}{8\pi} = P = \text{const}, \tag{4.7}$$

$$\rho V_x \boldsymbol{V}_\perp - \frac{B_x}{4\pi}\boldsymbol{B}_\perp = \boldsymbol{G} = \text{const}, \tag{4.8}$$

$$B_x \boldsymbol{V}_\perp - V_x \boldsymbol{B}_\perp = \boldsymbol{F} = \text{const}. \tag{4.9}$$

$$\rho\left(\frac{V^2}{2} + w\right)V_x + \frac{B^2}{4\pi}V_x = \mathcal{E} \tag{4.10}$$

These equations are algebraic, that is, if we find some solution it will remain constant in all space, for all x, unless MHD is broken somewhere. The equations have a discrete set of solutions, which means that there is no continuous transition from one set to another: ideal MHD is not capable of describing of structure continuously changing with x. Any transition from one set to another should be discontinuous.

4.2 Discontinuities

One way of breaking down MHD is to allow situations where the plasma variable change abruptly, that is, say $\rho(x<0) \neq \rho(x>0)$, while both are constant. In this case the variable is not determined at $x=0$. In fact, we have to allow such solutions in MHD since magnetohydrodynamics is unable to describe small-scale variations. On the other hand, (4.6)-(4.8) are nothing but the mass and momentum conservation laws, while (4.9) is simply a manifestation of the potentiality of the electric field in the time-dependent case, so that these equations have to be valid even in the case of abrupt changes.

Let us now rewrite (4.6)-(4.9) as follows:

$$J[V_x] + [p] + \left[\frac{B_\perp^2}{8\pi}\right] = 0, \tag{4.11}$$

$$J[\boldsymbol{V}_\perp] = \frac{B_x}{4\pi}[\boldsymbol{B}_\perp], \tag{4.12}$$

$$B_x[\boldsymbol{V}_\perp] = [V_x \boldsymbol{B}_\perp], \tag{4.13}$$

$$J[\frac{V^2}{2} + w] + [\frac{B^2 V_x}{4\pi}] - [\frac{B_x(\boldsymbol{V}\cdot\boldsymbol{B})}{4\pi}] = 0 \tag{4.14}$$

where $[A] \equiv A_2 - A_1 = A(x>0) - A(x<0)$, and $J = \rho_1 V_{1x} = \rho_2 V_{2x}$. In what follows we also assume that the state equation is polytropic on both sides of the discontinuity, with the same polytropic index:

$$w_1 = \frac{\gamma p_1}{(\gamma-1)\rho_1}, \quad w_2 = \frac{\gamma p_2}{(\gamma-1)\rho_2} \tag{4.15}$$

4.2.1 No-flow discontinuities

Let us first consider the case when $J=0$, which means $V_x = 0$. In this case

$$B_x[\boldsymbol{B}_\perp] = 0, \tag{4.16}$$

$$B_x[\boldsymbol{V}_\perp] = 0, \tag{4.17}$$

$$[p + \frac{B_\perp^2}{2}] = 0 \tag{4.18}$$

a) If $B_x \neq 0$ then $[\boldsymbol{V}_\perp] = 0$, $[\boldsymbol{B}_\perp] = 0$, $[p] = 0$, and only the density may be different on both sides, $[\rho] \neq 0$. Thus, the only difference in the plasma state on the both sides of the discontinuity is the difference in density (and temperature, as the pressure should be the same). By choosing the appropriate reference frame, moving along the discontinuity, we can always make $\boldsymbol{V}_\perp = 0$, so that there no flows at all, and everything is static. This is a *contact discontinuity* and it is the least interesting among all possible MHD discontinuities.

b) If $B_x = 0$ then it is possible that $[\boldsymbol{V}_\perp] \neq 0$ and $[\boldsymbol{B}_\perp] \neq 0$, while $[p + B_\perp^2/8\pi] = 0$. In addition to different densities at the both sides, the two plasmas are in a relative motion along the discontinuity. This is a *tangential discontinuity.*

4.2.2 Alfven (rotational) discontinuity

The situation changes drastically when $J \neq 0$, that is, there is a plasma flow across the discontinuity. Let us consider first the case where $[\rho] = 0$, that is, the density does not changes across the discontinuity. This immediately means $[V_x] = 0$ too, so that $V_x = \text{const}$ and from (4.12)-(4.13) we get

$$\rho V_x[\boldsymbol{V}_\perp] = \frac{B_x}{4\pi}[\boldsymbol{B}_\perp], \tag{4.19}$$

$$B_x[\boldsymbol{V}_\perp] = V_x[\boldsymbol{B}_\perp], \tag{4.20}$$

which means $V_x^2 = B_x^2/4\pi\rho$, that is, the velocity of the plasma is equal to the intermediate (Alfen) wave velocity. Hence, the discontinuity is called an *Alfven discontinuity.* Since in this structure the magnetic field rotates while its magnitude does not change (the velocity rotates too), it is also called a *rotational discontinuity.*

4.3 Shocks

The last discontinuity $J \neq 0$ and $[\rho] \neq 0$ is called a *shock* (explained below) and is the most important, therefore we devote a separate section to it. For the reasons which well be explained later we shall assume $\rho_2/\rho_1 > 1$, and $V_x > 0$, so that $V_{1x} > V_{2x}$. It is easy to show that $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{B}_1$, and $\boldsymbol{B}_2$ are in the same plane. We choose this plane as $x-z$ plane and the reference frame so that $\boldsymbol{V}_{1\perp} = 0$. Accordingly, $B_{1y} = B_{2y} = 0$, and $V_{2y} = 0$.

In what follows we shall assume that all variables at $x < 0$ (*upstream*, index 1) are known, and we are seeking to express all variables at $x > 0$ (*downstream*, index 2) with the use of known ones. Let us write again the conservation laws. Mass conservation:

$$\rho_1 V_{1x} = \rho_2 V_{2x} \tag{4.21}$$

Conservation of the x-component of the momentum:

$$\rho_1 V_{1x}^2 + p_1 + \frac{B_{1z}^2}{8\pi} = \rho_2 V_{2x}^2 + p_2 + \frac{B_{2z}^2}{8\pi} \tag{4.22}$$

Conservation of the z-component of the momentum:

$$-\frac{B_x B_{1z}}{4\pi} = \rho_2 V_{2x} V_{2z} - \frac{B_x B_{2z}}{4\pi} \tag{4.23}$$

Continuation of E_y:

$$cE_y = V_{1x}B_{1z} = V_{2x}B_{2z} - V_{2z}B_x \tag{4.24}$$

Conservation of energy

$$\rho_1 V_{1x}(\frac{V_{1x}^2}{2}+w_1)+\frac{cE_yB_{1z}}{4\pi}=\rho_2 V_{2x}(\frac{V_{2x}^2+V_{2z}^2}{2}+w_2)+\frac{cE_yB_{2z}}{4\pi} \tag{4.25}$$

Using (4.21) eq. (4.25) can be rewritten as follows:

$$\frac{V_{1x}^2}{2}+w_1+\frac{B_{1z}^2}{4\pi\rho_1}=\frac{V_{2x}^2+V_{2z}^2}{2}+w_2+\frac{B_{1z}B_{2z}}{4\pi\rho_1} \tag{4.26}$$

Eq. (4.24) gives

$$V_{2z}=\frac{B_x}{4\pi\rho_1 V_{1x}}(B_{2z}-B_{1z}) \tag{4.27}$$

Assuming the same state equation for the upstream and downstream, one has $w=\gamma p/(\gamma-1)\rho$, which gives the energy equation in the form

$$\frac{V_{1x}^2}{2}+\frac{\gamma p_1}{(\gamma-1)\rho_1}+\frac{B_{1z}^2}{4\pi\rho_1}=\frac{V_{2x}^2+V_{2z}^2}{2}+\frac{\gamma p_2}{(\gamma-1)\rho_2}+\frac{B_{1z}B_{2z}}{4\pi\rho_1} \tag{4.28}$$

Expressing the pressure from the pressure equation

$$p_2=\rho_1 V_{1x}^2+p_1+\frac{B_{1z}^2}{8\pi}-\rho_2 V_{2x}^2-\frac{B_{2z}^2}{8\pi} \tag{4.29}$$

and substituting into the energy equation one has

$$\frac{V_{1x}^2}{2}+\frac{\gamma p_1}{(\gamma-1)\rho_1}+\frac{B_{1z}^2}{4\pi\rho_1}=\frac{V_{2x}^2+V_{2z}^2}{2} \\ +\frac{B_{1z}B_{2z}}{4\pi\rho_1}+\frac{\gamma}{(\gamma-1)\rho_2}\left(\rho_1 V_{1x}^2-\rho_2 V_{2x}^2+\frac{B_{1z}^2-B_{2z}^2}{8\pi}+p_1\right) \tag{4.30}$$

We shall now introduce some notation which has a direct physical sense. Let $V_{1x}=V_u$, $B_x=B_u\cos\theta$ and $B_{1z}=B_u\sin\theta$, where B_u is the total upstream magnetic field, and θ is the angle between the upstream magnetic field vector and the shock *normal*. We define also the upstream Alfven velocity as $v_A^2=B_u^2/4\pi\rho_1$, and the *Alfvenic Mach number* as

$$M=V_{1x}/v_A \tag{4.31}$$

We further define

$$\beta=\frac{8\pi p_1}{B_u^2} \tag{4.32}$$

$$N=\frac{\rho_2}{\rho_1},\quad V=\frac{V_{1x}}{V_{2x}}=\frac{1}{N} \tag{4.33}$$

$$R=\frac{B_{2z}}{B_{1z}} \tag{4.34}$$

$$p_2=Pp_1N^\gamma,\quad w_2=Pw_1N^{\gamma-1} \tag{4.35}$$

With these definitions we have

$$1+\frac{\beta+\sin^2\theta}{2M^2}=\frac{1}{N}+\frac{P\beta N^\gamma+R^2\sin^2\theta}{2M^2} \tag{4.36}$$

$$u=\frac{V_{2z}}{V_u}=\frac{\sin\theta\cos\theta(R-1)}{M^2} \tag{4.37}$$

$$\left(\frac{\cos^2\theta}{M^2}-\frac{1}{N}\right)R=\left(\frac{\cos^2\theta}{M^2}-1\right) \tag{4.38}$$

$$T=\frac{P\beta N^\gamma}{2M^2}=(1-\frac{1}{N})+\frac{\beta}{2M^2}-\frac{(R^2-1)\sin^2\theta}{2M^2} \tag{4.39}$$

$$\frac{1}{2}+\frac{\gamma\beta}{2(\gamma-1)M^2}+\frac{\sin^2\theta}{M^2}=\frac{1+N^2u^2}{2N^2}+\frac{T\gamma}{(\gamma-1)N}+\frac{R\sin^2\theta}{M^2} \tag{4.40}$$

Thus, we reduced our problem to finding $N=\rho_2/\rho_1$, $R=B_{2z}/B_{1z}$, and P, as functions of M, θ, and β. It is easy to express everything in terms of N:

$$T=\frac{P\beta N^\gamma}{2M^2}=\left(1-\frac{1}{N}\right)+\frac{\beta}{2M^2}-\frac{(R^2-1)\sin^2\theta}{2M^2} \tag{4.41}$$

$$\frac{1}{2}+\frac{\gamma\beta}{2M^2(\gamma-1)}+\frac{\sin^2\theta}{M^2}=\frac{1}{2N^2}+\frac{u^2}{2}+\frac{R\sin^2\theta}{M^2} \tag{4.42}$$

$$+\left(\frac{\gamma}{N(\gamma-1)}\right)\left[\left(1-\frac{1}{N}\right)+\frac{\beta}{2M^2}-\frac{(R^2-1)\sin^2\theta}{2M^2}\right]$$

$$R=\frac{N(M^2-\cos^2\theta)}{M^2-N\cos^2\theta} \tag{4.43}$$

$$u=\frac{\sin\theta\cos\theta(N-1)}{M^2-N\cos^2\theta} \tag{4.44}$$

If $M\to\infty$ the density compression and the magnetic compression remain finite:

$$N\to R\to\frac{\gamma+1}{\gamma-1} \tag{4.45}$$

$$\frac{P}{M^2}\to\frac{4}{\beta(\gamma+1)}\left(\frac{\gamma+1}{\gamma-1}\right)^\gamma \tag{4.46}$$

It is convenient to write down as follows:

$$f(V,R)=\frac{(1-V^2)}{2}+\frac{\gamma\beta(1-V)}{2M^2(\gamma-1)}+\frac{\sin^2\theta(1-R)}{M^2} \tag{4.47}$$

$$-\frac{\sin^2\theta\cos^2\theta(R-1)^2}{2M^4}-\frac{V(1-V)\gamma}{(\gamma-1)}+\frac{V(R^2-1)\gamma\sin^2\theta}{2M^2(\gamma-1)}=0$$

$$g(V,R)=R(M^2V-\cos^2\theta)-(M^2-\cos^2\theta)=0 \tag{4.48}$$

$$\frac{R-1}{M^2}=\frac{1-V}{M^2V-\cos^2\theta} \tag{4.49}$$

$$\tag{4.50}$$

In what follows we consider only the simplest cases, leaving more detailed analysis for the advanced course or self-studies.

Parallel shock, $\theta = 0$. In this case $\sin\theta = 0$, $R = 1$, and $u = 0$, and one has

$$1 + \frac{\beta}{2M^2} = \frac{1}{N} + \frac{P\beta N^\gamma}{2M^2} \tag{4.51}$$

$$\frac{1}{2} + \frac{\gamma\beta}{2(\gamma-1)M^2} = \frac{1}{2N^2} + \frac{P\gamma\beta N^{\gamma-1}}{2(\gamma-1)M^2} \tag{4.52}$$

$$\frac{1}{2} + \frac{\gamma\beta}{2M^2(\gamma-1)} = \frac{1}{2N^2} + \left(\frac{\gamma}{\gamma-1}\right)\left[\left(\frac{1}{N} - \frac{1}{N^2}\right) + \frac{\beta}{2M^2N}\right] \tag{4.53}$$

It is easy to find

$$N = \frac{M^2(\gamma+1)}{M^2(\gamma-1) + \beta\gamma} \tag{4.54}$$

In order that this solution be $N > 1$ we have to require that

$$M^2 > \gamma\beta/2 \Rightarrow V_{1x}^2 > v_s^2 = \gamma p_1/\rho_1. \tag{4.55}$$

This relation means that the upstream velocity of the plasma flow should exceed the sound velocity. This is exactly the condition for a simple gasdynamical shock formation, and is seems quite reasonable since the magnetic field does not affect plasma motion at all. Yet we have to explain why $N > 1$ was required. It appears (we are not going to prove that in the course) that in this case entropy is increasing as the plasma flows across the shock, in accordance with the second thermodynamics law. In the opposite case, $N < 1$, the plasma entropy would decrease, which is not allowed. Yet, the condition (4.55) is not correct. The correct one is either $V_{1x} > v_F$ or $v_I > V_{1x} > v_{SL}$. This is because the true MHD waves are the fast, intermediate, and slow waves, and not the sound wave.

Perpendicular shock, $\theta = 90°$. In this case $\cos\theta = 0$, $u = 0$, $R = N$, and momentum and energy conservation give together

$$\frac{1}{2} + \frac{\gamma\beta}{2M^2(\gamma-1)} + \frac{1}{M^2} = \frac{1}{2N^2} + \left(\frac{\gamma}{N(\gamma-1)}\right)\left[(1 - \frac{1}{N}) + \frac{\beta}{2M^2} - \frac{(N-1)}{2M^2}\right] + \frac{N}{M^2} \tag{4.56}$$

$$\frac{1}{2} + \frac{\gamma\beta}{2M^2(\gamma-1)} + \frac{1}{M^2} = \frac{1}{2N^2} + \frac{N}{M^2} \tag{4.57}$$

$$+ \left(\frac{\gamma}{\gamma-1}\right)\left[\left(\frac{1}{N} - \frac{1}{N^2}\right) + \frac{\beta}{2M^2N} - \frac{(N^2-1)}{2M^2N}\right] \tag{4.58}$$

If instead, for simplicity, we restrict ourselves with the momentum conservation only and put $P = 1$ (which is incorrect, strictly speaking), then one gas

$$f(N) = \frac{1}{N} + \frac{\beta}{2M^2}N^\gamma + \frac{N^2}{2M^2} = 1 + \frac{\beta+1}{2M^2}. \tag{4.59}$$

A compressive solution, $N > 1$, exists, if

$$M^2 > 1 + \gamma\beta/2 \Rightarrow V_{1x} > \sqrt{v_A^2 + v_s^2}, \tag{4.60}$$

which means that the upstream plasma velocity should exceed the fast speed for perpendicular propagation.

4.4 Why shocks ?

Imagine a steady gas flow emerging from a source, and let this flow suddenly encounters an obstacle. The flow near the obstacle has to change in order to flow around. For the flow to re-arrange itself it should be affected in some way by the obstacle. In other words, those parts of the flow which should change must get information about the obstacle position, size, etc. The only way such information can propagate in the gas is by means of sound waves. That is, when the flow comes to the obstacle, the latter send sound waves backward (upstream) to affect those parts of the flow which are still far from the obstacle, to let them have sufficient time to re-arrange their velocity and density according to what should occur near the obstacle. The sound velocity is v_s relative to the flow. If the flow velocity is V, and sound has to propagate upstream (against the flow), its velocity relative to the obstacle would be $v_s - V$. It is obvious that the flow velocity near the obstacle is *subsonic*, $V < v_s$, so that sound waves can escape. If the flow velocity is subsonic everywhere, sound waves have no problem to reach the flow parts at any distance from the obstacle (that depends only on the time available) thus allowing the whole flow to re-arrange according to the obstacle requirements. As a result, in the steady state the gas parameters change smoothly from the source to the obstacle.

If, however, the gas flow is *supersonic*, $V > v_s$ far from the obstacle, those parts are not accessible by sound waves, since $v_s - V < 0$, which means that sound is dragged by the flow back to the obstacle. Yet the flow velocity at the obstacle itself must be subsonic, otherwise the gas could not flow around the obstacle. The only way to achieve that in hydrodynamics is to have a discontinuity, at which the gas velocity abruptly drops from a supersonic velocity to a subsonic one.

The same arguments work for MHD, except in this case MHD waves play the role of sound wave: whenever the plasma flow exceeds the velocity of the mode which is supposed to propagate information upstream (fast mode in our perpendicular case above), a shock has to form, where the plasma flow velocity drops from *super-magnetosonic* to *sub-magnetosonic*. As in the ordinary gas, this velocity drop is accompanied by the density and pressure (and temperature and entropy) increase. Thus, the primary role of a shock is a) to decelerate the flow from super-signal to sub-signal velocity, and b) convert the energy of the directed flow into thermal energy (in plasma also into magnetic energy, since the magnetic field also increases).

Of course, real shocks are not discontinuities but have some width, which is determined by microscopic processes at small spatial scales. At these scales gasdynamic or MHD approximations fail and have to refined or completed with something else (e.g. viscosity in the gas or resistivity in MHD) related to collisions (in the gas) or complex electromagnetic processes in collisionless plasmas. The latter, *collisionless* shocks play the very important role of the most efficient accelerators of charged particles in the universe.

4.5 Problems

PROBLEM 4.1. Consider a parallel shock $\boldsymbol{B}_\perp = 0$ and show that $\rho_2/\rho_1 \leq (\gamma+1)/(\gamma-1)$.

PROBLEM 4.2. What are the conditions for $B_{\perp 1} = 0$ but $B_{\perp 2} \neq 0$ in a shock ? For the opposite case ?

PROBLEM 4.3. Is it possible that the magnetic field decreases across a shock ?

PROBLEM 4.4. For an ideal gas entropy (per unit mass) $\propto p/\rho^\gamma$. Show that entropy does not change in small-amplitude waves but increases across a chock (consider parallel shocks).

Chapter 5

Two(multi)-fluid description

In this chapter we learn to improve our description of plasmas by analyzing motion of each plasma species instead of restricting ourselves to the single-fluid representation (MHD).

5.1 Basic equations

A plasma may consist of a number of species. Each species constitutes a separate (charged) fluid, so that we describe them by the following parameters: number density n_s, particle mass m_s, particle charge q_s, fluid velocity $\boldsymbol{V}_s$, and pressure p_s. For the two-fluid case (electron-ion plasma) $s = e, i$. In addition there are electric and magnetic fields present, which are related to the plasma.

Since each species is a fluid by itself it should be described by the equations similar to what we have already derived:

$$\frac{\partial}{\partial t} n_s + \boldsymbol{\nabla} \cdot (n_s \boldsymbol{V}_s) = 0, \tag{5.1}$$

$$n_s m_s \left(\frac{\partial}{\partial t} \boldsymbol{V}_s + (\boldsymbol{V}_s \cdot \boldsymbol{\nabla}) \boldsymbol{V}_s \right) = -\boldsymbol{\nabla} p_s + n_s q_s (\boldsymbol{E} + \boldsymbol{V}_s \times \boldsymbol{B}/c), \tag{5.2}$$

where we have included the electric force now, since each fluid is charged.

These equations should be completed with the state equations, like $p_s = p_s(n_s)$, and Maxwell equations in their full form withe the charge and current densities given as follows

$$\rho = \sum_s n_s q_s, \tag{5.3}$$

$$\boldsymbol{j} = \sum_s n_s q_s \boldsymbol{V}_s, \tag{5.4}$$

These charge and current enter the Maxwell equations, producing the electric and magnetic field, which, in turn, affect fluid motion and, therefore, produce the charge and current. Thus, the interaction between electrons and ions occurs via the self-consistent electric and magnetic fields, and the necessary bootstrap is achieved.

5.2 No magnetic field case

Let us assume now that there is neither ambient magnetic field nor self-consistent magnetic field arising due to plasma currents. The latter requires $\boldsymbol{j} = \sum_s n_s q_s \boldsymbol{V}_s = 0$. In the absence of magnetic fields the electric field is a potential field, $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$. Accordingly, the equations take the form

$$\partial_t n_s + \boldsymbol{\nabla}(n_s \boldsymbol{V}_s) = 0 \tag{5.5}$$

$$m_s\left(\partial_t \boldsymbol{V}_s + (\boldsymbol{V}_s \cdot \boldsymbol{\nabla})\boldsymbol{V}_s\right) = -q_s \boldsymbol{\nabla}\phi - \frac{1}{n_s}\boldsymbol{\nabla} p_s \tag{5.6}$$

$$\triangle\phi = -4\pi \sum_s n_s q_s \tag{5.7}$$

5.2.1 Small-amplitude (linear) waves

We start with the equilibrium $\boldsymbol{V}_s = 0$, $n_s = n_{0s}$, $\phi = 0$ and allow for small perturbations $\delta\boldsymbol{V}_s$, δn_s, $\delta\phi$. Retaining only the linear terms, one has

$$\partial_t \delta n_s + \boldsymbol{\nabla}(n_{0s}\delta\boldsymbol{V}_s) = 0 \tag{5.8}$$

$$m_s \partial_t \delta\boldsymbol{V}_s = -q_s \boldsymbol{\nabla}\delta\phi - m_s c_s^2 \boldsymbol{\nabla}\delta n_s \tag{5.9}$$

$$\triangle\phi = -4\pi \sum_s \delta n_s q_s \tag{5.10}$$

$$m_s c_s^2 = \left(\frac{dp_s}{dn_s}\right)_{n_s = n_{0s}} \tag{5.11}$$

As usual, we are seeking for solutions $\propto \exp(i\boldsymbol{k}\boldsymbol{r} - i\omega t)$, which gives

$$\boldsymbol{k}\cdot\delta\boldsymbol{V}_s = \frac{\omega\delta n_s}{n_{0s}} \tag{5.12}$$

$$\omega m_s \delta\boldsymbol{V}_s = \boldsymbol{k}[q_s\delta\phi + m_s c_s^2 \delta_s] \tag{5.13}$$

One can see that $\delta\boldsymbol{V}_s \propto \boldsymbol{k}$. Multiplying by $\boldsymbol{k}$ and substituting the first equation to the second and further to the Poisson equation, one finds the dispersion relation in the form

$$1 = \sum_s \frac{\omega_{ps}^2}{\omega^2 - k^2 c_s^2} \tag{5.14}$$

$$\omega_{ps}^2 = \frac{4\pi q_s^2 n_{0s}}{m_s} \tag{5.15}$$

The dispersion relation can be also written in the form

$$k^2 = \sum_s \frac{\omega_{ps}^2}{v_{ph}^2 - c_s^2} \tag{5.16}$$

where $v_{ph} = \omega/k$ is the phase velocity of the wave.

Let us consider in more detail the case of electron-proton plasma, where $q_i = -q_e = e$, $n_{0i} = n_{0e} = n_0$, $m_i/m_e \approx 2000 \gg 1$. Assuming $T_e \sim T_i$ and $\gamma_e \sim \gamma_i$ we have $c_i^2/c_e^2 \sim m_e/m_i \ll 1$. We have also $\omega_{pi^2}/\omega_{pe}^2 = m_i/m_e \ll 1$. The dispersion relation now takes the form

$$k^2 = \frac{\omega_{pe}^2}{v_{ph}^2 - c_e^2} + \frac{\omega_{pi}^2}{v_{ph}^2 - c_i^2} \tag{5.17}$$

The dispersion relation is a biquadratic equation for ω as a function of k and can be easily solved. However, it is instructive to analyze it in various limits. Treating k^2 as a function of v_{ph}^2, we see that $k^2 > 0$ in two regions: a) $v_{ph}^2 > c_e^2$ and b) $c_i^2 < v_{ph}^2 < c_a^2 < c_e^2$. Here by c_a we denote the phase velocity where $k^2 = 0$.

Case a) $v_{ph}^2 > c_e^2$. If $v_{ph} > c_e$ one has approximately

$$1 \approx \frac{m_e}{m_i}\frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega^2 - k^2 c_e^2} \tag{5.18}$$

Since the first term is $O(m_e/m_i)$ relative to the second term, we get

$$\omega^2 = \omega_{pe}^2 + k^2 c_e^2 \tag{5.19}$$

This is the Langmuir wave. In the longwavelength limit it gives $\omega = \omega_{pe}$.

Case b) $c_i^2 < v_{ph}^2 < c_a^2 < c_e^2$. In this case one has

$$1 = \frac{\omega_{pi}^2}{\omega^2 - k^2 c_i^2} - \frac{\omega_{pi}^2}{k^2 c_e^2 (m_e/m_i)} \tag{5.20}$$

$$\omega^2 = k^2 c_i^2 + \frac{k^2 c_a^2}{1 + k^2 c_a^2/\omega_{pi}^2} \tag{5.21}$$

where $c_a^2 = c_e^2(m_e/m_i) = \gamma_e T_e/m_i$ is the *ion sound speed.* The wave is called ion sound or ion-acoustic wave. For $k \to 0$ one gets

$$\omega^2 = k^2 \frac{\gamma_i T_i + \gamma_e T_e}{m_i} \tag{5.22}$$

if the ions are cold, $T_i = 0$, the pressure is due to the electrons and the inertia is due to the ions. The electrons behave adiabatically in the electric field, following the potential as it were static. Expanding for small k one obtains also the weak dispersion in the form

$$\omega = k c_a \left(1 - \frac{k^2 c_a^2}{\omega_{pi}^2}\right) \tag{5.23}$$

In the high frequency (large k) limit one gets

$$\omega^2 = k^2 c_i^2 \tag{5.24}$$

Case c) $v_{ph}^2 \ll c_i^2, c_e^2$. In this case one gets

$$k^2 = -\sum_s \frac{\omega_{ps}^2}{c_s^2} = -\frac{1}{r_D^2} \tag{5.25}$$

which is nothing but the Debye screening.

5.3 Nonlinear ion-acoustic waves

Let us consider one-dimensional electrostatic waves neglecting the electron mass and the ion pressure:

$$\partial_t n_i + \partial_x (n_i V_i) = 0 \tag{5.26}$$
$$m_i(\partial_t V_i + V_i \partial_x V_i) = -e\partial_x \phi \tag{5.27}$$
$$0 = en_e \partial_x \phi - \partial_x p_e \tag{5.28}$$
$$\partial_x^2 \phi = -4\pi e(n_i - n_e) \tag{5.29}$$

The equation for electrons simply means

$$n_e = n_0 \exp(e\phi/T_e) \tag{5.30}$$

5.3.1 Stationary waves

Stationary waves are those which depend on $x - Vt$, where $V = \text{const}$. It is always possible to switch to the wave frame, where the dependence on t disappears. In this case we get

$$\frac{m_i V_i^2}{2} + e\phi = \frac{m_i V_0^2}{2} \tag{5.31}$$
$$n_i = \frac{n_0 V_0}{V_i} = \frac{n_0}{\sqrt{1 - 2e\phi/m_i V_0^2}} \tag{5.32}$$
$$\partial_x^2 \phi = -4\pi n_0 e \left[\frac{1}{\sqrt{1 - 2e\phi/m_i V_0^2}} - \exp(e\phi/T_e) \right] \tag{5.33}$$

Let us introduce the following physical variables

$$\xi = \frac{x}{r_D}, \quad \psi = \frac{e\phi}{T_e}, \quad M^2 = \frac{m_i V_0^2}{T_e}, \quad r_D^2 = \frac{T_e}{4\pi n_0 e^2} \tag{5.34}$$

the equation for the potential can be integrated once to the following pseudopotential form

$$\frac{1}{2}\left(\frac{d\psi}{d\xi}\right)^2 + U(\psi) = E = \text{const} \tag{5.35}$$
$$U(\psi) = 1 - e^{\psi} + M^2 \left(1 - \sqrt{1 - \frac{2\psi}{M^2}}\right) \tag{5.36}$$

The "energy" $E = 0$ corresponds to the solution which has $d\psi/dx = 0$ for $\psi = 0$, that is, the solution which passes through the equilibrium point. A nontrivial solution of this kind exists only if the equilibrium is a maximum (unstable equilibrium), that is,

$$\left(\frac{d^2U}{d\psi^2}\right)_{\psi=0} = \frac{1}{M^2} - 1 < 0 \to M > 1 \tag{5.37}$$

This is not the only condition. The other one is the existence of the point $\psi_m < 2M^2$ such that $U(\psi_m) = 0$. If both conditions are satisfied, there exists a soliton solution which has $\psi = 0$ at $x = \pm\infty$ and $\psi = \psi_m$ in one point between the two asymptotic.

Let us analyze in more detail a weakly nonlinear solution with $M - 1 = \epsilon \ll 1$. Then also $\psi \ll 1$ and we have (up to the third order)

$$U(\psi) = -\epsilon\psi^2 + \frac{2}{3}\psi^3, \tag{5.38}$$

$$\psi_m = \sqrt{3\epsilon/2} \tag{5.39}$$

5.4 Time-dependent nonlinear waves

Let us introduce the Lagrange coordinates

$$d\xi = n_i(dx - V dt), \quad \tau = t \tag{5.40}$$

one finds

$$\partial_x = n_i\partial_\xi \tag{5.41}$$

$$\partial_t = \partial_\tau - n_i V\partial_\xi \tag{5.42}$$

$$\partial_t + V\partial_x = \partial_\tau \tag{5.43}$$

The equations for the nonlinear waves take the form

$$\partial_\tau q = \partial_\xi V \tag{5.44}$$

$$\partial_\tau V = -\frac{e}{m_i} n_i \partial_\xi \phi \tag{5.45}$$

$$n_i\partial_\xi(n_i\partial_\xi\phi)\phi = -4\pi e(n_i - n_0 e^{e\phi/T}) \tag{5.46}$$

$$q = 1/n_i \tag{5.47}$$

$$\partial_\tau^2 q = -\frac{e}{m}\partial_\xi n_i \partial_\xi \phi \tag{5.48}$$

5.5 Reduction to MHD

Somehow we should be able to derive the MHD equations from the two-fluid equations, otherwise there would be internal inconsistency in the plasma theory. We start with mentioning one of the conditions of MHD, namely, quasi-neutrality, which means $\rho = \sum_s n_s q_s = 0$ (actually, negligible). Notice now that summing the equations of motion for all species we get

$$\sum_s n_s m_s \left(\frac{\partial}{\partial t}\boldsymbol{V}_s + (\boldsymbol{V}_s \cdot \boldsymbol{\nabla})\boldsymbol{V}_s\right) = -\boldsymbol{\nabla}(\sum_s p_s) + \rho\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}/c, \tag{5.49}$$

and we should drop the electric term in view of the above condition. The right hand side now looks as it should be if we notice that $p = \sum_s p_s$ is the total plasma pressure. The left hand side still does not look like it was in the MHD case. Before we proceed further we rewrite the obtained equation in another form (see (2.12)):

$$\frac{\partial}{\partial t}\sum_s(n_s m_s V_{si}) + \frac{\partial}{\partial x_j}\sum_s(n_s m_s V_{si} V_{sj}) = -\frac{\partial}{\partial x_i}p + \frac{1}{c}\epsilon_{ijk} j_j B_k \tag{5.50}$$

The quantity $\rho_m = \sum_s n_s m_s$ is nothing but the mass density, and $\sum_s (n_s m_s V_{si})$ is nothing but the momentum density, thus the mass flow velocity should be defined as

$$\boldsymbol{V} = \sum_s (n_s m_s \boldsymbol{V}_s)/\rho, \qquad \rho = \sum_s n_s m_s. \tag{5.51}$$

Now the sum of the (5.1) multiplied by m_s gives the mass flow continuity equation (2.5).

We have yet to make the term $\sum_s (n_s m_s V_{si} V_{sj})$ look like $\rho V_i V_j$, if possible. Here we have to be more explicit. Let us write down the obtained relations ($s = 1, 2$ instead of i, e here for convenience):

$$n_1 m_1 \boldsymbol{V}_1 + n_2 m_2 \boldsymbol{V}_2 = \rho \boldsymbol{V}, \tag{5.52}$$

$$n_1 q_1 \boldsymbol{V}_1 + n_2 q_2 \boldsymbol{V}_2 = \boldsymbol{j}, \tag{5.53}$$

from which it is easy to find

$$\boldsymbol{V}_1 = \frac{\rho g_2 \boldsymbol{V} - \boldsymbol{j}}{n_1 m_1 (g_2 - g_1)} \tag{5.54}$$

$$\boldsymbol{V}_2 = \frac{\rho g_1 \boldsymbol{V} - \boldsymbol{j}}{n_2 m_2 (g_1 - g_2)} \tag{5.55}$$

where $g_s = q_s/m_s$. Proceeding further, we find

$$\boldsymbol{V}_1 = \boldsymbol{V} - \frac{\boldsymbol{j}}{n_1 m_1 (g_2 - g_1)},$$

where we have taken into account that $n_1 m_1 g_1 + n_2 m_2 g_2 = 0$. After substitution and some algebra we have

$$\sum_s (n_s m_s V_{si} V_{sj}) = \rho V_i V_j + \frac{\rho j_i j_j}{n_1 n_2 m_1 m_2 (g_1 - g_2)^2},$$

which is not exactly what we are looking for. This means that MHD is an approximation where we neglect the jj term relative to VV term. Let us have a close look at this negligence when $q_1 = -q_2 = e$, $m_1 = m_i \gg m_2 = m_e$, $n_1 = n_2$ (electron-ion plasma). In this case $g_1 \ll |g_2| = e/m_e$, and the jj term takes the following form

$$\frac{\rho j_i j_j m_e}{n^2 e^2 m_i}$$

and can be neglected when

$$j \ll neV\sqrt{m_i/m_e},$$

that is, when the current is not extremely strong.

5.6 Generalized Ohm's law

Let us focus on the electron-ion plasma where $m_e \ll m_i$. For simplicity we also assume quasineutrality, $n_e = n_i = n$, which happens when motion is slow and electrons can easily adjust their density to neutralize ions. Using $\boldsymbol{j} = ne(\boldsymbol{V}_i - \boldsymbol{V}_e)$ we substitute $\boldsymbol{V}_e = \boldsymbol{V}_i - \boldsymbol{j}/ne$ into the electron equation of motion and get

$$m_e \frac{d\boldsymbol{V}_e}{dt} = -e(\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}/c) + \frac{1}{nc}\boldsymbol{j} \times \boldsymbol{B} - \frac{1}{n}\,\mathsf{grad}\, p_e \tag{5.56}$$

or

$$\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}/c = \frac{1}{nec}\boldsymbol{j} \times \boldsymbol{B} - \frac{1}{en}\,\mathsf{grad}\, p_e - \frac{m_e}{e}\frac{d\boldsymbol{V}_e}{dt}. \tag{5.57}$$

The expression (5.57) is known as the generalized Ohm's law. If there was no right hand side (zero electron mass, cold electrons, weak currents) it would become $\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}/c = 0$. Since in this limit the single-fluid velocity $\boldsymbol{V} = \boldsymbol{V}_i$, this is nothing but the Ohm's law in ideal MHD. The terms in the right hand side of (5.57) modify the Ohm's law, adding the Hall term (first), the pressure induced electric field (second) and the electron inertia term (third).

5.7 Nonlinear magnetosonic waves

5.7.1 Time-dependent waves

For simplicity let us consider a wave which propagates in the direction perpendicular to the magnetic field in a cold plasmas, that is, let $\boldsymbol{B} = (0, 0, B(x,t))$, $\boldsymbol{v} = (V_x(x,t), V_y(x,t), 0)$, and $p_e = p_i = 0$. We shall also assume quasineutrality $n_e = n_i = n$ and neglect the displacement current. Then one has also $V_{ex} = V_{ix} = V_x$ and

$$\partial_t n + \partial_x(nV) = 0 \tag{5.58}$$

$$m_i(\partial_t V + V\partial_x V) = eE_x + \frac{e}{c}V_{iy}B \tag{5.59}$$

$$m_e(\partial_t V + V\partial_x V) = -eE_x - \frac{e}{c}V_{ey}B \tag{5.60}$$

$$m_i(\partial_t V_{iy} + V\partial_x V_{iy}) = eE_y - \frac{e}{c}VB \tag{5.61}$$

$$m_e(\partial_t V_{ey} + V\partial_x V_{ey}) = -eE_y + \frac{e}{c}VB \tag{5.62}$$

$$\partial_t B = c\partial_x E_y \tag{5.63}$$

$$\frac{4\pi}{c}n(V_{iy} - V_{ey}) = -\partial_x B \tag{5.64}$$

From (5.61) and (5.62) we find $V_{iy}/V_{ey} = -m_e/m_i$ and, therefore,

$$V_{ey} = -\frac{c}{4\pi n}\partial_x B \tag{5.65}$$

Eqs. (5.59) and (5.60) give

$$m_i(\partial_t V + V\partial_x)V = -\frac{1}{n}\partial_x\left(\frac{B^2}{8\pi}\right) \tag{5.66}$$

Further,

$$E_y = -\frac{VB}{c} + m_e(\partial_t + V\partial_x)\frac{c}{4\pi n}\partial_x B \tag{5.67}$$

$$(\partial_t + V\partial_x)n = -n\partial_x V \tag{5.68}$$

$$(\partial_t + V\partial_x)B = -B\partial_x V + \frac{m_e c^2}{4\pi}\partial_x(\partial_t + V\partial_x)\frac{1}{n}\partial_x B \tag{5.69}$$

Introducing the coordinates

$$d\xi = n(dx - Vdt), \quad \tau = t \tag{5.70}$$

one gets

$$\partial_\tau q = \partial_\xi V, \quad q = 1/n \tag{5.71}$$

$$q\partial_\tau B = -B\partial_\xi V + \frac{m_e c^2}{4\pi}\partial_\tau \partial_\xi^2 B \tag{5.72}$$

$$m_i \partial_\tau V = -\partial_\xi \left(\frac{B^2}{8\pi}\right) \tag{5.73}$$

and eventually

$$\partial_\tau^2 q = -\partial_\xi^2 \left(\frac{B^2}{8\pi}\right) \tag{5.74}$$

$$\partial_\tau(qB) = \frac{m_e c^2}{4\pi}\partial_\tau \partial_\xi^2 B \tag{5.75}$$

$$qB = \frac{m_e c^2}{4\pi}\partial_\xi^2 B + q_0 B_0 \tag{5.76}$$

Following the earlier procedure, one can derive KdV for weakly nonlinear waves.

5.7.2 Magnetosonic soliton

Let us consider stationary perpendicular waves for which $\partial_t = 0$. In this case

$$cE_y = -VB + \frac{m_e cV}{e^2}\frac{d}{dx}\frac{c^2}{4\pi n}\frac{d}{dx}B = \text{const} \tag{5.77}$$

$$nm_i V^2 + \frac{B^2}{8\pi} = n_0 m_i V_0^2 + \frac{B_0^2}{8\pi} \tag{5.78}$$

$$nV = n_0 V_0 \tag{5.79}$$

The first equation is rewritten as follows, using $nV = n_0 V_0$:

$$cE_y = -VB + \frac{m_e c n_0 V_0}{4\pi e^2 n}\frac{d}{dx}\frac{c^2}{4\pi n}\frac{d}{dx}B \tag{5.80}$$

Using normalized variables

$$B/B_0 \to B, \quad V/V_0 \to V, \omega_{pe}x/c \to x, \tag{5.81}$$

$$V_0/v_A = M, \quad v_A^2 = \frac{B_0^2}{4\pi n_0 m_i} \tag{5.82}$$

$$d\xi = n dx/n_0 \tag{5.83}$$

one has

$$\frac{d^2}{d\xi^2}B = VB - F, \quad F = \text{const} \tag{5.84}$$

$$V = 1 + \frac{1 - B^2}{2M^2} \tag{5.85}$$

Let us look for a solution where $\frac{d^2}{d\xi^2}B = 0$ in the reference point, then $F = 1$ and

$$\frac{d^2}{d\xi^2}B = f(B) = B\left(1 + \frac{1 - B^2}{2M^2}\right) - 1 \tag{5.86}$$

In a more traditional way, let us assume that $dB/d\xi = 0$ and $d^2B/d\xi^2 = 0$ when $\xi \to -\infty$. The standard method of solving the equation is to represent it in a pseudo-potential form

$$\frac{d^2}{d\xi^2}B = -\frac{dU}{dB} \tag{5.87}$$

$$U(B) = (B - 1) - \frac{(B^2 - 1)}{2}\left(1 + \frac{1}{2M^2}\right) + \frac{(B^4 - 1)}{8M^2} \tag{5.88}$$

Arbitrarily we choose $U(B = 1) = 0$. Then one has

$$\frac{1}{2}\left(\frac{dB}{d\xi}\right)^2 + U(B) = 0 \tag{5.89}$$

A solution exists only if the point $B = 1$ is a maximum, that is,

$$\frac{d^2U}{dB^2}_{B=1} = \frac{1}{M^2} - 1 < 0 \to M > 1 \tag{5.90}$$

Since $U(B \to \pm\infty) \to \infty$, there will be always another $B = B_m$, for which $U(B_m) = 0$. This would correspond to a soliton solution.

5.8 Problems

Problem 5.1. Let $q_2 = -q_1$ and $m_2 = m_1$. Derive single-fluid equations from two-fluid ones in the assumption $n_2 = n_1$ (quasineutrality).

Problem 5.2. Derive generalized Ohm's law for a quasineutral electron-positron plasma.

Problem 5.3. Write down two-fluid equations when there is friction (momentum transfer) between electrons and ions.

Problem 5.4. Derive the Hall-MHD equations substituting the ideal MHD Ohm's law with

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}/c = \frac{1}{nec}\boldsymbol{j} \times \boldsymbol{B}$$

Problem 5.5. Treat electrons as massless fluid and derive corresponding HD equations for quasineutral motion.

Chapter 6

Waves in dispersive media

In this chapter we learn basics of the general theory of waves in dispersive media.

6.1 Maxwell equations for waves

Whatever medium it is, if propagating waves are of electromagnetic nature they should be described by the Maxwell equations (2.27)-(2.28). We already know that the two other equations are just constraints, and once satisfied would be satisfied forever. The vacuum electromagnetic waves are discovered when $\boldsymbol{j} = 0$. It is rather obvious that the "only" influence of the medium is via the current $\boldsymbol{j}$. In general, this current should include also the magnetization current, and can be nonzero even without applying external fields, like in ferromagnets. Although the theory can be developed for these cases too, for simplicity we shall limit ourselves with the situations where the current is *induced* by the fields themselves, that is, in the absence of the fields (except constant homogeneous fields for which $\mathsf{rot} = 0$ and $(\partial/\partial t) = 0$) the current $\boldsymbol{j} = 0$.

We start again with an equilibrium, where the only possible field is a constant homogeneous magnetic field $\boldsymbol{B}_0 = \text{const}$. Let us assume that the equilibrium is perturbed, that is, there appear time- and space-varying electric and magnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$. These fields induce current $\boldsymbol{j}$ which we shall consider as being a function of $\boldsymbol{E}$ (since $\boldsymbol{B}$ and $\boldsymbol{E}$ are closely related it is always possible). In general, $\boldsymbol{j}$ may be a nonlinear function of $\boldsymbol{E}$. However, if the fields are weak, we can assume that the induced current is weak too and is linearly dependent on the electric field. In a most general way this can be written as follows

$$j_i(\boldsymbol{r}, t) = \int \Lambda_{ij}(\boldsymbol{r}, \boldsymbol{r}', t, t') E_j(\boldsymbol{r}', t') d\boldsymbol{r}' dt'. \tag{6.1}$$

Thus, current "here and now" depends on the electric field in "there and then". The simplest form of this relation implies $\Lambda_{ij} = \sigma\delta_{ij}\delta(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t')$ and results in the ordinary Ohm's law $\boldsymbol{j} = \sigma\boldsymbol{E}$. The function Λ is determined by the features of the medium and does not depend on $\boldsymbol{E}$.

If the equilibrium is homogeneous and time-stationary, the integration kernel should depend only on $\boldsymbol{r} - \boldsymbol{r}'$ and $t - t'$. In this case one may Fourier-transform

(6.1),

$$j_i(\boldsymbol{r}, t) = \int j_i(\boldsymbol{k}, \omega) \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] d\boldsymbol{k} d\omega, \tag{6.2}$$

to obtain

$$j_i(\boldsymbol{k}, \omega) = \sigma_{ij}(\boldsymbol{k}, \omega) E_j(\boldsymbol{k}, \omega). \tag{6.3}$$

where σ_{ij} is the *conductivity tensor*.

We proceed by Fourier-transforming the Maxwell equations (2.27)-(2.28) which gives $\boldsymbol{B} = c\boldsymbol{k} \times \boldsymbol{E}/\omega$ (we do not denote differently Fourier components) and eventually

$$D_{ij} E_j \equiv \left[\frac{k^2 c^2}{\omega^2} \delta_{ij} - \frac{k_i k_j c^2}{\omega^2} - \epsilon_{ij} \right] E_j = 0, \tag{6.4}$$

where we have defined the *dielectric tensor*

$$\epsilon_{ij} = \delta_{ij} + \frac{4\pi i}{\omega} \sigma_{ij}. \tag{6.5}$$

Expression (6.4) is a set of homogeneous equations. In order to have nonzero solutions for E_i we have to require

$$D(\boldsymbol{k}, \omega) \equiv \det ||D_{ij}|| = 0, \tag{6.6}$$

which is known as a *dispersion relation*. The very existence of the dispersion relation means that the frequency ω of the wave and the wave vector $\boldsymbol{k}$ are not independent. This is quite natural. Indeed, even in the vacuum the two are related as $\omega = kc$. All effects related to the medium are in the dielectric tensor ϵ_{ij} (or in σ_{ij}).

Thus, the only perturbations which can survive in a dispersive medium should be $\propto \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega(\boldsymbol{k})t)]$, where we emphasize the dependence of the frequency on the wave vector.

6.2 Wave amplitude, velocity etc.

Once $D = 0$ the rank of the matrix D_{ij} reduces to 2, which means that we are left with only two independent equations for three components of the electric field. As usual, it means that one of these components can be chosen arbitrarily, while the two others will be expressed in terms of the chosen one. Warning: the above statement is not precise and not any component can be chosen as an arbitrary one in all cases. Once we have chosen one component we shall refer to it as a *wave amplitude*. This is rather imprecise and a more rigorous definition would be based on some physical concept, like *wave energy*. This will be considered in an advanced section below.

Let $e_i(\boldsymbol{k})$ be the *unit* vector corresponding to the wave with the wave vector $\boldsymbol{k}$. It is called a polarization vector. The electric field corresponding to this wave can be written as $E_i = a(\boldsymbol{k}) e_i(\boldsymbol{k})$, where a is the amplitude. The general solution for the wave (Maxwell) equations in the dispersive medium is

$$E_i(\boldsymbol{r}, t) = \int a(\boldsymbol{k}) e_i(\boldsymbol{k}) \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega(\boldsymbol{k})t)] d\boldsymbol{k}. \tag{6.7}$$

There is no already integration over ω since it is determined by $\boldsymbol{k}$. Let initially, at $t = 0$, the electric field $\boldsymbol{E} = \boldsymbol{E}_0(\boldsymbol{r})$. Then

$$E_{i0}(\boldsymbol{r}) = \int a(\boldsymbol{k})e_i(\boldsymbol{k}) \exp[i\boldsymbol{k}\cdot\boldsymbol{r}]d\boldsymbol{k}. \tag{6.8}$$

and the amplitudes may be found by inverse Fourier-transform:

$$a(\boldsymbol{k})e_i(\boldsymbol{k}) = (2\pi)^{-3}\int E_{i0}(\boldsymbol{r}) \exp[-i\boldsymbol{k}\cdot\boldsymbol{r}]d\boldsymbol{r}. \tag{6.9}$$

Further substitution into (6.7) would give the electric field at all times.

The form $\Phi = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t$ is the wave *phase*. The Considering Φ as an instantaneous function of $\boldsymbol{r}$, the normal to the constant phase surface (*wave front*) would be given by $\hat{n} = \mathsf{grad}\,\Phi/|\,\mathsf{grad}\,\Phi| = \boldsymbol{k}/k$. It is clear that the constant phase surfaces in our case are planes perpendicular to $\boldsymbol{k}$, hence the wave is a *plane wave*. Let us consider the same constant phase surfaces= $\Phi = \Phi_0$ at moments t and $t + dt$, and let ds be the distance between the two planes along the normal. Then one has

$$\boldsymbol{k}\cdot(ds\hat{\boldsymbol{n}}) - \omega dt = 0,$$

so that the velocity of the constant phase surface, the so-called *phase velocity* is

$$\boldsymbol{v}_{ph} = \frac{ds}{dt}\hat{\boldsymbol{n}} = \frac{\omega}{k}\hat{\boldsymbol{k}}. \tag{6.10}$$

The phase velocity describes only the phase propagation and is not related to the energy transfer. Thus, it is not limited from above and can exceed the light speed. It is worth noting that $n = kc/\omega = c/v_{ph}$ is the refraction index.

In order to analyze propagation of physical quantities we have to consider a *wave packet*. Let us assume that initial perturbation exists only in a finite space region of the size $|\Delta\boldsymbol{r}|$. The uncertainty principle (or Fourier-transform properties) immediately tells us that the amplitude $\boldsymbol{a}(\boldsymbol{k})$ should be large only in the vicinity of some $\boldsymbol{k}_0$: for $|\boldsymbol{k} - \boldsymbol{k}_0| > \Delta\boldsymbol{k}$ the amplitude is negligible. Here $|\Delta\boldsymbol{k}\cdot\Delta\boldsymbol{r}| \sim 1$. Let us assume that in this range $\omega(\boldsymbol{k})$ is a sufficiently slowly varying function, so that we can Taylor expand

$$\omega(\boldsymbol{k}) = \omega_0 + \kappa_i\frac{\partial\omega}{\partial k_i} + \tfrac{1}{2}\kappa_i\kappa_j\frac{\partial^2\omega}{\partial k_i k_j}, \tag{6.11}$$

where $\boldsymbol{\kappa} = \boldsymbol{k} - \boldsymbol{k}_0$, $\omega_0 = \omega(\boldsymbol{k}_0)$, and all derivatives are taken at $\boldsymbol{k} = \boldsymbol{k}_0$. Then the wave packet at moment t would take the form

$$\begin{aligned} E_i(\boldsymbol{r},t) &= \int a(\boldsymbol{k})e_i(\boldsymbol{k}) \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega(\boldsymbol{k})t)]d\boldsymbol{k} \\ &= \exp[i(\boldsymbol{k}_0\cdot\boldsymbol{r} - \omega_0 t)]\int a(\boldsymbol{\kappa})e_i(\boldsymbol{\kappa}) \exp[i\boldsymbol{\kappa}\cdot(\boldsymbol{r} - \boldsymbol{v}_g t)] \\ &\cdot \exp[-(i/2)\kappa_i\kappa_j\frac{\partial^2\omega}{\partial k_i k_j}]d\boldsymbol{\kappa} \approx \exp(i\Phi)E_{i0}(\boldsymbol{r} - \boldsymbol{v}_g t), \end{aligned} \tag{6.12}$$

where the *group velocity* $\boldsymbol{v}_g = (d\omega/d\boldsymbol{k})$, that is, $v_{gi} = (\partial\omega/\partial k_i)$, and in the last line we neglected the second derivative term in the exponent. Thus, the velocity $\boldsymbol{v}_g$ approximately corresponds to the motion of the initial profile. It can be shown that the second derivative term describes the variation of the profile shape. It should be sufficiently small (shape does not change much when the whole profile moves) in order that the group velocity be of physical sense.

6.3 Wave energy

Let us define

$$D_i(\boldsymbol{r},t) = E_i(\boldsymbol{r},t) + 4\pi \int_{-\infty}^{t} j_i(\boldsymbol{r},t')dt', \tag{6.13}$$

where j_i is the *internal* current, that is, the current produced by the same particles which are moving in the wave. In general, external (not related directly to the wave) currents can be present, that is, $\boldsymbol{j} = \boldsymbol{j}_{int} + \boldsymbol{j}_{ext}$. Since

$$j_i = \int_{-\infty}^{t} dt' \int d\boldsymbol{r}' \sigma_{ij}(\boldsymbol{r}-\boldsymbol{r}', t-t')E_j(\boldsymbol{r}',t') \tag{6.14}$$

and, respectively,

$$j_i(\omega,\boldsymbol{k}) = \sigma_{ij}(\omega,\boldsymbol{k})E_j(\omega,\boldsymbol{k}) \tag{6.15}$$

we can write

$$D_i = \int_{-\infty}^{t} dt' \int d\boldsymbol{r}' \epsilon_{ij}(\boldsymbol{r}-\boldsymbol{r}', t-t')E_i(\boldsymbol{r}',t') \tag{6.16}$$

and

$$D_i(\omega,\boldsymbol{k}) = \epsilon_{ij}(\omega,\boldsymbol{k})E_j(\omega,\boldsymbol{k}) \tag{6.17}$$

where

$$\epsilon_{ij}(\omega,\boldsymbol{k}) = \delta_{ij} + \frac{4\pi i}{\omega}\sigma_{ij}(\omega,\boldsymbol{k}) \tag{6.18}$$

The current equation takes the form

$$\frac{\partial \boldsymbol{D}}{\partial t} = c\,\mathsf{rot}\,\boldsymbol{B} + 4\pi \boldsymbol{j}_{ext}. \tag{6.19}$$

Multiplying this equation by $\boldsymbol{E}$, the induction equation by $\boldsymbol{B}$ and summing up we get

$$\begin{aligned}
&\frac{1}{4\pi}\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} + \frac{1}{4\pi}\boldsymbol{B}\cdot\frac{\partial \boldsymbol{B}}{\partial t} \\
&= \frac{c}{4\pi}(\boldsymbol{E}\cdot\mathsf{rot}\,\boldsymbol{B} - \boldsymbol{B}\cdot\mathsf{rot}\,\boldsymbol{E}) + 4\pi\boldsymbol{E}\cdot\boldsymbol{j}_{ext} \\
&= -\frac{c}{4\pi}\,\mathsf{div}(\boldsymbol{E}\times\boldsymbol{B}) + 4\pi\boldsymbol{E}\cdot\boldsymbol{j}_{ext}.
\end{aligned} \tag{6.20}$$

Averaging this equation over a volume large enough relative to the typical length of variations (much larger than the wavelength) we get

$$\begin{aligned}
&\frac{1}{V}\int_V \left[\frac{1}{4\pi}\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} + \frac{\partial}{\partial t}\frac{B^2}{8\pi}\right] dV \\
&= -\int_V \frac{c}{4\pi}\,\mathsf{div}(\boldsymbol{E}\times\boldsymbol{B})dV + 4\pi\boldsymbol{E}\cdot\boldsymbol{j}_{ext} \\
&= -\frac{1}{V}\int_S \frac{c}{4\pi}\,\mathsf{div}(\boldsymbol{E}\times\boldsymbol{B})\cdot d\boldsymbol{S} + 4\pi\boldsymbol{E}\cdot\boldsymbol{j}_{ext}
\end{aligned} \tag{6.21}$$

The first term in the right hand side is the energy flux outward from the volume. The second term is the work done by the wave electric field on the external currents. Therefore, the left hand side should be interpreted as the rate of change of the wave energy. Let $\bar{\omega}$ be the typical frequency of the wave. The wave energy change implies the wave amplitude change. For a monochromatic wave

$$E_i(\boldsymbol{r}, t) = E_i \exp(i\boldsymbol{kr} - i\bar{\omega}t) + E_i^* \exp(-i\boldsymbol{kr} + i\bar{\omega}t) \tag{6.22}$$

where $E_i = \text{const}$. There is no energy change of a monochromatic wave. In order to allow energy variation we have to assume that E_i depends on time weakly, so that $(1/E)(\partial E/\partial t) \ll \bar{\omega}$. In other words,

$$E_i(\boldsymbol{r}, t) = \int d\omega E_i(\omega) E_i \exp(i\boldsymbol{kr} - i\omega t) + E_i^* \exp(-i\boldsymbol{kr} + i\omega t) \tag{6.23}$$

where $|\omega - \bar{\omega}| \ll \bar{\omega}$. Since we are interested in slow variations, in what following we have to analyze

$$\begin{aligned} \langle \frac{1}{4\pi}\boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \frac{1}{4\pi}\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \rangle \\ = \frac{1}{T}\int dt \left(\frac{1}{4\pi}\boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \frac{1}{4\pi}\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \right) \end{aligned} \tag{6.24}$$

where $T \gg 1/\bar{\omega}$.

We shall perform calculations in a more general way. Let

$$E_i(\boldsymbol{r}, t) = \int d\omega d\boldsymbol{k} \left(E_i(\omega, \boldsymbol{k}) e^{i(\boldsymbol{kr} - \omega t)} + E_i^*(\omega, \boldsymbol{k}) e^{-i(\boldsymbol{kr} - \omega t)} \right) \tag{6.25}$$

where $\omega > 0$. Then

$$\begin{aligned} D_i(\boldsymbol{r}, t) = \int d\omega d\boldsymbol{k} \big(& \epsilon_{ij}(\omega, \boldsymbol{k}) E_j(\omega, \boldsymbol{k}) e^{i(\boldsymbol{kr} - \omega t)} \\ & + \epsilon_{ij}^*(\omega, \boldsymbol{k}) E_j^*(\omega, \boldsymbol{k}) e^{-i(\boldsymbol{kr} - \omega t)} \big) \end{aligned} \tag{6.26}$$

This expression uses the relation

$$\epsilon_{ij}(-\omega, -\boldsymbol{k}) = \epsilon_{ij}^*(\omega, \boldsymbol{k})$$

Now

$$\begin{aligned} \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} = \int & d\omega d\omega' d\boldsymbol{k} d\boldsymbol{k}' \left(E_i(\omega, \boldsymbol{k}) e^{i(\boldsymbol{kr} - \omega t)} + E_i^*(\omega, \boldsymbol{k}) e^{-i(\boldsymbol{kr} - \omega t)} \right) \\ & \cdot i\omega' \Big(\epsilon_{ij}(\omega', \boldsymbol{k}') E_j(\omega', \boldsymbol{k}') e^{i(\boldsymbol{k}'\boldsymbol{r} - \omega' t)} \\ & - \epsilon_{ij}^*(\omega', \boldsymbol{k}') E_j^*(\omega', \boldsymbol{k}') e^{-i(\boldsymbol{k}'\boldsymbol{r} - \omega' t)} \Big) \end{aligned} \tag{6.27}$$

When averaging over large volume this results in $< e^{i(\boldsymbol{k} - \boldsymbol{k}')\boldsymbol{r}} >= \delta(\boldsymbol{k} - \boldsymbol{k}')$. Let us write

$$\epsilon_{ij} = \epsilon_{ij}^H + \epsilon_{ij}^A \tag{6.28}$$

where the Hermitian part satisfies

$$\epsilon^H_{ij} = {\epsilon^{H*}}_{ji} \tag{6.29}$$

and the anti-Hermitian part satisfies

$$\epsilon^A_{ij} = -{\epsilon^{A*}}_{ji} \tag{6.30}$$

so that

$$\begin{aligned}
< \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} > = i & \int d\omega d\omega' d\boldsymbol{k} (E_j(\omega, \boldsymbol{k}) E^*_i(\omega', \boldsymbol{k}) \\
& \cdot e^{-i(\omega-\omega')t} (\omega' \epsilon_{ij}(\omega', \boldsymbol{k}) - \omega \epsilon^*_{ji}(\omega, \boldsymbol{k})) \\
= i & \int d\omega d\omega' d\boldsymbol{k} (E_j(\omega, \boldsymbol{k}) E^*_i(\omega', \boldsymbol{k}) \\
& \cdot e^{-i(\omega-\omega')t} (\omega' \epsilon^H_{ij}(\omega', \boldsymbol{k}) - \omega \epsilon^H_{ij}(\omega, \boldsymbol{k})) \\
- & \int d\omega d\omega' d\boldsymbol{k} (E_j(\omega, \boldsymbol{k}) E^*_i(\omega', \boldsymbol{k}) \\
& \cdot e^{-i(\omega-\omega')t} (\omega' \epsilon^A_{ij}(\omega', \boldsymbol{k}) + \omega \epsilon^A_{ij}(\omega, \boldsymbol{k}))
\end{aligned} \tag{6.31}$$

Here we dropped the terms with $\exp[\pm i(\omega + \omega')t]$ since, when averaging over $T \gg 1/\omega$, these fast oscillation terms vanish. On the other hand, the terms with $\exp[\pm i(\omega - \omega')t]$ survive when $|\omega - \omega'|T \ll 1$, that is, $\omega' \approx \omega$. It is worth emphasizing that exact equality is not required since the variation time $1/|\omega - \omega'|$ is larger than the averaging time. In this case we can Taylor expand:

$$\omega' \epsilon^H_{ij}(\omega', \boldsymbol{k}) - \omega \epsilon^H_{ij}(\omega, \boldsymbol{k}) = (\omega' - \omega) \frac{\partial}{\partial \omega} \left(\omega \epsilon^H_{ij}(\omega, \boldsymbol{k}) \right)$$

and

$$\omega' \epsilon^A_{ij}(\omega', \boldsymbol{k}) + \omega \epsilon^A_{ij}(\omega, \boldsymbol{k}) = 2\omega \epsilon^A_{ij}(\omega, \boldsymbol{k})$$

The term with the anti-Hermitian ϵ^A is responsible for the intrinsic nonstationarity of the way amplitude. In the thermodynamic equilibrium it describes the natural dissipation of the wave energy. We shall not consider it here. For the rest of the expression notice that

$$i(\omega' - \omega) e^{-i(\omega-\omega')t} = \frac{d}{dt} e^{-i(\omega-\omega')t}$$

and therefore (restoring all integrations)

$$< \int_V dV \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} >= \frac{d}{dt} \int d\omega d\boldsymbol{k} (E_j(\omega, \boldsymbol{k}) E^*_i(\omega, \boldsymbol{k}) \frac{\partial}{\partial \omega} \left(\omega \epsilon^H_{ij}(\omega, \boldsymbol{k}) \right) \tag{6.32}$$

For a wave with the dispersion relation $\omega = \omega(\boldsymbol{k})$ one has

$$E_j(\omega, \boldsymbol{k}) E^*_i(\omega, \boldsymbol{k}) = E_j(\omega(\boldsymbol{k}), \boldsymbol{k}) E^*_i(\omega(\boldsymbol{k}), \boldsymbol{k}) \delta(\omega - \omega(\boldsymbol{k})) \tag{6.33}$$

so that we get

$$< \int_V dV \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} >= \frac{d}{dt} \int d\boldsymbol{k} (E_j(\omega, \boldsymbol{k}) E^*_i(\omega, \boldsymbol{k}) \frac{\partial}{\partial \omega} \left(\omega \epsilon^H_{ij}(\omega, \boldsymbol{k}) \right) \tag{6.34}$$

where now ω is not independent by has to be found from the dispersion relation.

Now we see that the wave energy can be identified as

$$U = \frac{\partial}{\partial\omega}\left(\omega\epsilon_{ij}^H(\omega,\boldsymbol{k})\right)\frac{E_j(\omega,\boldsymbol{k})E_i^*(\omega,\boldsymbol{k})}{4\pi} + \frac{B_i(\omega,\boldsymbol{k})B_i^*(\omega,\boldsymbol{k})}{4\pi} \tag{6.35}$$

where $\omega = \omega(\boldsymbol{k})$. Let us now take into account that

$$B_i = \varepsilon_{ijk}n_jE_k, \qquad n_i = \frac{k_ic}{\omega}$$

so that

$$\frac{B_m(\omega,\boldsymbol{k})B_m^*(\omega,\boldsymbol{k})}{4\pi} = (n^2\delta_{ij} - n_in_j)E_iE_j^*$$

and

$$\begin{aligned} U &= [n^2\delta_{ij} - n_in_j + \frac{\partial}{\partial\omega}(\omega\epsilon_{ij}^H)]\frac{E_jE_i^*}{4\pi} \\ &= [n^2\delta_{ij} - n_in_j - \epsilon_{ij}^H + \frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega^2\epsilon_{ij}^H)]\frac{E_jE_i^*}{4\pi} \\ &= \frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega^2\epsilon_{ij}^H)\frac{E_jE_i^*}{4\pi} \end{aligned} \tag{6.36}$$

since $(n^2\delta_{ij} - n_in_j + \epsilon_{ij}^H)E_j = 0$ because of the dispersion relation and only the Hermitian part of ϵ_{ij} is implied. If we now represent the wave electric field as $E_i = E\hat{e}_i$, where E is the wave amplitude, and $\hat{e}_i$ is the wave polarization (unit vector), one gets

$$U_{\boldsymbol{k}} = \frac{\hat{e}_i^*\hat{e}_j}{\omega}\frac{\partial}{\partial\omega}(\omega^2\epsilon_{ij}^H)\frac{|E|^2}{4\pi} \tag{6.37}$$

6.4 Problems

PROBLEM 6.1. Given the initial profile of $A(x, t = 0) = A_0\exp(-x^2/2L^2)$ and the dispersion relation $\omega = \pm kv$, find the wave profile at $t > 0$.

PROBLEM 6.2. Same profile but ω does not depend on k.

PROBLEM 6.3. Same but $\omega = \pm ak^2$.

PROBLEM 6.4. Same but $\omega^2 = k^2v^2/(1 + k^2d^2)$.

PROBLEM 6.5. Given $n(\omega)$ find the group velocity.

PROBLEM 6.6. What are the conditions on $(d\omega/dk)$ and $(d^2\omega/dk^2)$ when group velocity has physical sense ?

Problem 6.7. In what conditions initial discontinuity propagates as a discontinuity ?

Chapter 7

Waves in two-fluid hydrodynamics

In this chapter we apply the theory of waves in dispersive media to the cold two-fluid hydrodynamics.

7.1 Dispersion relation

Let us consider a plasma consisting of two fluids, electrons and ions. We shall denote species with index $s = e, i$. For simplicity we consider cold species only, so the the corresponding hydrodynamical equations read

$$\frac{\partial n_s}{\partial t} + \mathsf{div}(n_s \boldsymbol{V}_s) = 0, \tag{7.1}$$

$$n_s m_s \left(\frac{\partial \boldsymbol{V}_s}{\partial t} + (\boldsymbol{V}_s \cdot \boldsymbol{\nabla}) \boldsymbol{V}_s \right) = q_s n_s (\boldsymbol{E} + \boldsymbol{V}_s \times \boldsymbol{B}/c). \tag{7.2}$$

In the equilibrium $n_s = n_{s0}$, $\boldsymbol{V}_s = 0$, $\boldsymbol{E} = 0$, and $\boldsymbol{B} = \boldsymbol{B}_0$. As usual, we write down the Fourier-transformed linearized equations for deviations from the equilibrium, $\tilde{n}_s(\boldsymbol{k}, \omega)$, $\tilde{\boldsymbol{V}}_s(\boldsymbol{k}, \omega)$, $\tilde{\boldsymbol{E}}(\boldsymbol{k}, \omega)$ (we do not need perturbations of the magnetic field):

$$\tilde{n}_s = n_{s0} \boldsymbol{k} \cdot \tilde{\boldsymbol{V}}_s / \omega, \tag{7.3}$$

$$- i\omega \tilde{\boldsymbol{V}}_s = g_s (\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{V}}_s \times \boldsymbol{B}_0 / c), \tag{7.4}$$

where $g_s = q_s/m_s$. Our ultimate goal is to find the current $\tilde{\boldsymbol{j}} = \sum_s n_{s0} q_s \tilde{\boldsymbol{V}}_s$, so that we do not need (7.3). In order to solve (7.4) let us choose coordinates so that $\boldsymbol{B}_0 = B_0 \hat{z}$ and rewrite the equations as follows:

$$- i\omega \tilde{V}_{sz} = g_s \tilde{E}_z, \tag{7.5}$$

$$- i\omega \tilde{V}_{sx} - \Omega_s \tilde{V}_{sy} = g_s \tilde{E}_x, \tag{7.6}$$

$$- i\omega \tilde{V}_{sy} + \Omega_s \tilde{V}_{sx} = g_s \tilde{E}_y, \tag{7.7}$$

where $\Omega_s = g_s B_0 / c$ is the species *gyrofrequency*. Equation (7.5) is immediately solved:

$$\tilde{V}_{sz} = \frac{i g_s}{\omega} \tilde{E}_z. \tag{7.8}$$

The other two components are most easily found if we define $\tilde{E}_l = \tilde{E}_x + il\tilde{E}_y$, and $\tilde{V}_l = \tilde{V}_x + il\tilde{V}_y$, where $l = \pm 1$. Then (7.6) and (7.7) give

$$-i(\omega - l\Omega_s)\tilde{V}_{sl} = g_s\tilde{E}_l,$$

and, eventually,

$$\tilde{V}_{sl} = \frac{ig_s}{\omega - l\Omega_s}\tilde{E}_l. \tag{7.9}$$

Proceeding further, one has

$$\begin{aligned} \tilde{V}_{sx} + il\tilde{V}_{sy} &= \frac{ig_s}{\omega^2 - \Omega_s^2}(\omega + l\Omega_s)\tilde{E}_l \\ &= \frac{ig_s}{\omega^2 - \Omega_s^2}\left[(\omega\tilde{E}_x + i\Omega_s\tilde{E}_y) + l(\omega\tilde{E}_y - i\Omega_s\tilde{E}_x)\right] \end{aligned} \tag{7.10}$$

so that eventually we get

$$\tilde{V}_{sx} = \frac{ig_s\omega}{\omega^2 - \Omega_s^2}\tilde{E}_x - \frac{g_s\Omega_s}{\omega^2 - \Omega_s^2}\tilde{E}_y, \tag{7.11}$$

$$\tilde{V}_{sy} = \frac{g_s\Omega_s}{\omega^2 - \Omega_s^2}\tilde{E}_x + \frac{ig_s\omega}{\omega^2 - \Omega_s^2}\tilde{E}_y. \tag{7.12}$$

Respectively, the current will take the form

$$j_x = \left(\sum_s \frac{ig_s n_{s0} q_s\omega}{\omega^2 - \Omega_s^2}\right)\tilde{E}_x - \left(\sum_s \frac{g_s n_{s0} q_s\Omega_s}{\omega^2 - \Omega_s^2}\right)\tilde{E}_y, \tag{7.13}$$

$$j_y = \left(\sum_s \frac{g_s n_{s0} q_s\Omega_s}{\omega^2 - \Omega_s^2}\right)\tilde{E}_x + \left(\sum_s \frac{ig_s n_{s0} q_s\omega}{\omega^2 - \Omega_s^2}\right)\tilde{E}_y, \tag{7.14}$$

$$j_z = \left(\sum_s \frac{ig_s n_{s0} q_s}{\omega}\right)\tilde{E}_z. \tag{7.15}$$

Now, using the definitions of σ_{ij} and ϵ_{ij} we can arrive at the following dielectric tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_\perp & iG & 0 \\ -iG & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix}, \tag{7.16}$$

$$\epsilon_\perp = 1 + \left(\sum_s \frac{\omega_{ps}^2}{\Omega_s^2 - \omega^2}\right), \tag{7.17}$$

$$G = \left(\sum_s \frac{\omega_{ps}^2\Omega_s}{\omega(\Omega_s^2 - \omega^2)}\right), \tag{7.18}$$

$$\epsilon_\parallel = 1 - \left(\sum_s \frac{\omega_{ps}^2}{\omega^2}\right), \tag{7.19}$$

$$\omega_{ps}^2 = \frac{4\pi n_{s0} q_s^2}{m_s}. \tag{7.20}$$

It is convenient to choose coordinates so that $\boldsymbol{k} = (k_\perp, 0, k_\parallel) = k(\sin\theta, 0, \cos\theta)$, where $\theta = \widehat{\boldsymbol{B}_0\boldsymbol{k}}$ is the angle between the wave vector and the equilibrium magnetic field. Then the dispersion equation will take the following form

$$\begin{pmatrix} n^2\cos^2\theta - \epsilon_\perp & -iG & -n^2\sin\theta\cos\theta \\ iG & n^2 - \epsilon_\perp & 0 \\ -n^2\sin\theta\cos\theta & 0 & n^2\sin^2\theta - \epsilon_\parallel \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0. \tag{7.21}$$

Before we analyze the complete dispersion relation (determinant) let us consider special cases.

7.2 Unmagnetized plasma

In this case $B_0 = 0$ and, therefore, $\Omega_s = 0$ and $G = 0$. Moreover, $\epsilon_\perp = \epsilon_\parallel$. Since the dielectric tensor is isotropic, $\epsilon_{ij} = \epsilon_\parallel \delta_{ij}$, dispersion relations cannot depend on the direction, and we may choose $\theta = 0$. We get two identical dispersion relations (for $E_x \neq 0$ and for $E_y \neq 0$) which give

$$\omega^2 = \omega_p^2 + k^2c^2, \qquad \omega_p^2 = \sum_s \omega_{ps}^2, \tag{7.22}$$

with the phase velocity $v_{ph} = c\sqrt{1 + \omega_p^2/k^2c^2} > c$ and group velocity $v_g = c/v_{ph}$. The wave is transverse, $\boldsymbol{E} \perp \boldsymbol{k}$ and purely electromagnetic. There is no density perturbation. The lowest possible frequency is ω_p.

The third dispersion relation is for $E_z \neq 0$ and gives $\omega = \omega_p$. This longitudinal wave, $\boldsymbol{E} \parallel \boldsymbol{k}$ is the density perturbation waves and is called Langmuir wave.

7.3 Parallel propagation

In this case $\theta = 0$. The determinant is

$$D = \left((n^2 - \epsilon_\perp)^2 - G^2\right)\epsilon_\parallel = 0. \tag{7.23}$$

The solution $\epsilon_\parallel = 0$ corresponds to the Langmuir wave with $E_z \neq 0$. The two other correspond to $E_y = \pm iE_x$ (circular polarization) and $n^2 = \epsilon_\perp \pm G$:

$$n^2 = 1 + \frac{\omega_{pi}^2}{\omega(\Omega_i - \omega)} - \frac{\omega_{pe}^2}{\omega(|\Omega_e| + \omega)}, \tag{7.24}$$

$$n^2 = 1 - \frac{\omega_{pi}^2}{\omega(\omega + \Omega_i)} + \frac{\omega_{pe}^2}{\omega(|\Omega_e| - \omega)}. \tag{7.25}$$

For a simple electron-proton plasma $q_i = e$, $q_e = -e$, $m_i/m_e \approx 2000 \gg 1$. We have $|\Omega_e|/\Omega_i = m_i/m_e$, $\omega_{pe}/\omega_{pi} = \sqrt{m_i/m_e}$, and

$$\frac{\omega_{pe}^2}{|\Omega_e|} = \frac{\omega_{pi}^2}{\Omega_i}, \tag{7.26}$$

$$\frac{\omega_{pe}^2}{|\Omega_e^2|} = \frac{m_e}{m_i}\frac{\omega_{pi}^2}{\Omega_i^2} \tag{7.27}$$

Low frequencies. In the range $\omega \ll \Omega_i$ Taylor expansion gives for both modes

$$n^2 = 1 + \frac{\omega_{pi}^2}{\Omega_i^2}, \tag{7.28}$$

or (for the typical $\omega_{pi} \gg \Omega_i$) $\omega = kc\Omega_i/\omega_{pi} = kv_A$, where

$$v_A^2 = \frac{c^2\Omega_i^2}{\omega_{pi}^2} = \frac{B^2}{4\pi n_i m_i}.$$

High frequencies. In the range $\omega \gg |\Omega_e|$ Taylor expansion gives for both modes $n^2 = 1 + \omega_{pe}^2/\omega^2$, that is, electromagnetic modes in an unmagnetized plasma.

Intermediate frequencies. In the range $\Omega_i \ll \omega \ll |\Omega_e|$ one has only one mode

$$n^2 = \frac{\omega_{pe}^2}{\omega|\Omega_e|} = \frac{\omega_{pi}^2}{\omega|\Omega_i|}.$$

This is so-called *whistler* $\omega = k^2c^2\Omega_i/\omega_{pi}^2$. This is wave is strongly dispersive, $v_g \propto k$.

Analysis. Let us write

$$n_s^2 = 1 - \frac{\omega_{pi}^2}{\omega(\omega - s\Omega_i)} - \frac{\omega_{pe}^2}{\omega(\omega + s|\Omega_e|)} \tag{7.29}$$

$$= \frac{\omega(\omega - s\Omega_i)(\omega + s|\Omega_e|) - \omega_{pi}^2(\omega + s|\Omega_e|) - \omega_{pe}^2(\omega - s\Omega_i)}{\omega(\omega - s\Omega_i)(\omega + s|\Omega_e|)} \tag{7.30}$$

$$= \frac{\omega^2 + s\omega(|\Omega_e| - \Omega_i) - (\omega_{pe}^2 + \omega_{pi}^2 + \Omega_i|\Omega_e|)}{(\omega - s\Omega_i)(\omega + s|\Omega_e|)} \tag{7.31}$$

where $s = 1$. With the chosen ordering of frequencies, one can approximately write

$$n_s^2 = \frac{\omega^2 + s\omega|\Omega_e| - \omega_{pe}^2}{(\omega - s\Omega_i)(\omega + s|\Omega_e|)} \tag{7.32}$$

Consider now the limit $\omega \to \infty$:

$$n_s^2 = 1 \tag{7.33}$$

which is the regular electromagnetic wave in the vacuum.

Consider the limit $\omega \to 0$:

$$n_s^2 = \frac{\omega_{pe}^2 - s\omega|\Omega_e|}{\Omega_i|\Omega_e|} \tag{7.34}$$

$$= \frac{\omega_{pi}^2}{\Omega_i^2}\left(1 - \frac{s\omega\Omega_i}{\omega_{pi}^2}\right) \tag{7.35}$$

which gives

$$\omega = kv_A\left(1 + \frac{skc}{\omega_{pi}}\right) \tag{7.36}$$

Consider now $k = 0$ which gives

$$\omega = \omega_{pe} \tag{7.37}$$

Finally, consider $n^2 \to \infty$ which gives $\omega = \Omega_i$ for $s = 1$ and $\omega = |\Omega_e|$ for $s = -1$.

7.4 Perpendicular propagation

In this case $\cos\theta = 0$ and we have either

$$n^2 = \epsilon_\parallel$$

which is the electromagnetic wave like in the unmagnetized plasma, or

$$n^2 = \epsilon_\perp - \frac{G^2}{\epsilon_\perp}.$$

In the low frequency range, $\omega \ll \Omega_i$, we get $\omega = kc\Omega_i/\omega_{pi} = kv_A$,

In the whole energy range one has

$$n^2 = \frac{(\epsilon_\perp - G)(\epsilon_\perp + G)}{\epsilon_\perp} \tag{7.38}$$

where

$$\epsilon_\perp = 1 - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} \tag{7.39}$$

$$= \frac{(\omega^2 - \Omega_e^2)(\omega^2 - \Omega_i^2) - \omega_{pi}^2(\omega^2 - \Omega_e^2) - \omega_{pe}^2(\omega^2 - \Omega_i^2)}{(\omega^2 - \Omega_e^2)(\omega^2 - \Omega_i^2)} \tag{7.40}$$

$$= \frac{\omega^4 - \omega^2(\Omega_e^2 + \Omega_i^2 + \omega_{pi}^2 + \omega_{pe}^2) + (\Omega_e^2\Omega_i^2 + \omega_{pi}^2\Omega_e^2 + \omega_{pe}^2\Omega_i^2)}{(\omega^2 - \Omega_e^2)(\omega^2 - \Omega_i^2)} \tag{7.41}$$

$$\approx \frac{\omega^4 - \omega^2\omega_{pe}^2 + \omega_{pe}^2\Omega_i|\Omega_e|}{(\omega^2 - \Omega_e^2)(\omega^2 - \Omega_i^2)} \tag{7.42}$$

Therefore

$$n^2 \approx \frac{(\omega^2 - \omega_{pe}^2)^2}{\omega^4 - \omega^2\omega_{pe}^2 + \omega_{pe}^2\Omega_i|\Omega_e|} \tag{7.43}$$

Let us perform the same analysis as for the parallel propagation. Consider $\omega \to \infty$ which gives

$$n^2 = 1 \tag{7.44}$$

Consider $\omega \to 0$, then

$$n^2 \approx \frac{\omega_{pe}^2}{\Omega_i|\Omega_e| - \omega^2} \tag{7.45}$$

$$\Omega_i|\Omega_e| - \omega^2 = \frac{\omega_{pe}^2}{k^2c^2}\omega^2 \tag{7.46}$$

$$\omega^2 = \frac{k^2v_A^2}{1 + k^2c^2/\omega_{pe}^2} \tag{7.47}$$

Consider $k = 0$, then $\omega = \omega_{pe}$. Finally, consider $n \to \infty$, which gives

$$\omega^2 \approx \Omega_i|\Omega_e| \tag{7.48}$$

7.5 General properties of the dispersion relation

It is easy to see that in general case the dispersion relation for (7.21) takes the form

$$A(\omega)n^4 + B(\omega)n^2 + C(\omega) = 0, \tag{7.49}$$

where

$$A = \epsilon_\parallel \cos^2\theta + \epsilon_\perp \sin^2\theta, \tag{7.50}$$
$$B = G^2 \sin^2\theta - \epsilon_\perp(\epsilon_\parallel + A), \tag{7.51}$$
$$C = \epsilon_\parallel(\epsilon_\perp^2 - G^2). \tag{7.52}$$

Thus, there are two solutions for n^2, in general. It can be shown that these solutions are real. The regions where $n^2 < 0$ correspond to *non-transparency*: the corresponding mode does not propagate in this range.

7.6 Problems

PROBLEM 7.1. Derive dispersion relations for electrostatic waves propagating along the magnetic field taking into account the electron and ion pressure.

PROBLEM 7.2. Find cutoff frequencies ($k \to 0$) for parallel and perpendicular propagation.

PROBLEM 7.3. Show that waves become longitudinal, $\boldsymbol{E} \parallel \boldsymbol{k}$, when $n^2 \to \infty$ and find frequencies of these oscillations (resonance frequencies).

PROBLEM 7.4. Derive dispersion relations for an electron-positron plasma.

PROBLEM 7.5. Let a plasma consist of electrons and *two* ion species. Derive the dispersion relation for waves propagating in the direction parallel to the magnetic field. What is new ?

PROBLEM 7.6. An electromagnetic wave propagates in the vacuum in the direction perpendicular to the vacuum-plasma interface. There is no external magnetic field. The wave frequency $\omega < \omega_p$. Describe the electric field in the plasma.

PROBLEM 7.7. A linearly polarized electromagnetic wave of frequency $\omega \gg |\Omega_e|$ enters a column of plasma along the external magnetic field. Calculate the rotation of the electric field vector upon crossing the plasma.

PROBLEM 7.8. Describe the polarization of obliquely propagating waves.

PROBLEM 7.9. Let $m_e \to 0$. Derive the corresponding dispersion relations.

Chapter 8

Kinetic theory

In this chapter we learn a more detailed method of plasma description.

8.1 Distribution function

In the hydrodynamic description of plasma we forgot about different velocities of plasma particles and described it with the help of averaged quantities only: density, hydrodynamical velocity, and pressure. In this description the only reminder of the random (thermal) motion of particles was pressure. A more sophisticated description would give us information about different particle motion, at least at some level of averaging. The approach is in some sense similar to the hydrodynamical density approach. Indeed, density $n(\boldsymbol{r}, t)$ is nothing but the indication that the number of particles within the volume $d\mathcal{V} = d^3\boldsymbol{r}$ is $dN = n(\boldsymbol{r}, t)d\mathcal{V}$ at the moment t, if we properly average fast and small scale fluctuations. The last means that any *physically* infinitesimal volume should contain a large number of particles, and that the time averaging is over the time which is much larger than any time required for any microscopic relaxation process.

Following this principle, we consider the phase space $(\boldsymbol{r}, \boldsymbol{p})$ and define a phase space density, which is more often called *distribution function*, as follows: $dN = f(\boldsymbol{r}, \boldsymbol{p}, t)d^3\boldsymbol{r}d^3\boldsymbol{p}$ is a number of particles in a physically infinitesimal volume of the phase space.

In a more conservative way, f is often defined as a probability for a particle to be in the phase space volume $d^3\boldsymbol{r}d^3\boldsymbol{p}$, so that $\int f d^3\boldsymbol{r}d^3\boldsymbol{p} = 1$, when the integration is over the whole phase space. We shall use the first definition, except stated otherwise.

It is easy to see that the particle density and hydrodynamical velocity are related to the distribution function in a simple way:

$$n(\boldsymbol{r}, t) = \int f d^3\boldsymbol{p}, \tag{8.1}$$

$$n\boldsymbol{V}(\boldsymbol{r}, t) = \int \boldsymbol{v} f d^3\boldsymbol{p}. \tag{8.2}$$

In general, an integral of the kind $v_{i1} \dots v_{in} f d^3\boldsymbol{p}$ is called nth moment of the distribution function. Let us consider the second moment (for simplicity we restrict ourselves with nonrelativistic particles so that $p_i = mv_i$):

$$\begin{aligned}\int v_i v_j f d^3\boldsymbol{p} &= \int [(v_i - V_i) + V_i][(v_j - V_j) + V_j] f d^3\boldsymbol{p} \\ &= \int (v_i - V_i)(v_j - V_j) f d^3\boldsymbol{p} + nV_iV_i \\ &= p_{ij}/m + nV_iV_j,\end{aligned} \tag{8.3}$$

where p_{ij} is the *pressure tensor*. In the case of ideal gas $p_{ij} = p\delta_{ij}$, where $p/m = (1/3)\int v^2 f d^3\boldsymbol{p}$.

8.2 Kinetic equation

We need some tools for the description of the evolution of the distribution function. In order to do this we recall that a particle motion is determined by Hamiltonian dynamics, which means that the initial position $\boldsymbol{r}_0$ and momentum $\boldsymbol{p}_0$ completely determine, in principle, the particle future. In other words it is stated in the form of the Liouville theorem: the phase space volume does not change with time. The last statement means that the total time derivative of the distribution function along the trajectory vanishes:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0. \tag{8.4}$$

Taking into account that $\dot{\boldsymbol{r}} = \boldsymbol{v}$ and $\dot{\boldsymbol{p}} = \boldsymbol{F}$ (force), we get the kinetic Vlasov equation in the following form:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0. \tag{8.5}$$

For the nonrelativistic plasma $\boldsymbol{p} = m\boldsymbol{v}$, and $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c)$, so that eventually we get the equation in the form we will be using throughout:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{q}{m}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0. \tag{8.6}$$

If a plasma consists of several species s, Vlasov equation should be written for each distribution function f_s. The set should be completed with the Maxwell equations where

$$\rho = \sum_s q_s \int f_s d^3\boldsymbol{v}, \tag{8.7}$$

$$\boldsymbol{j} = \sum_s q_s \int \boldsymbol{v} f_s d^3\boldsymbol{v}. \tag{8.8}$$

Here we switched from $\boldsymbol{p}$ to $\boldsymbol{v}$ properly re-normalizing the distribution function.

8.3 Relation to hydrodynamics

Let us take zeroth moment of (8.6), that is, integrate it over $d^3\boldsymbol{v}$. We get

$$\begin{aligned}
&\int \frac{\partial f}{\partial t} d^3\boldsymbol{v} + \int v_i \frac{\partial f}{\partial x_i} d^3\boldsymbol{v} + \int \frac{q}{m}(E_i + \epsilon_{ijk} v_j B_k/c) \frac{\partial f}{\partial v_i} d^3\boldsymbol{v} \\
&= \frac{\partial}{\partial t}(\int f d^3\boldsymbol{v}) + \frac{\partial}{\partial x_i}(\int v_i f d^3\boldsymbol{v}) \\
&= \frac{\partial}{\partial t} n + \frac{\partial}{\partial x_i}(nV_i) = 0,
\end{aligned} \tag{8.9}$$

which is nothing but the continuity equation.

First moment will give, respectively,

$$\begin{aligned}
&m \int v_a \frac{\partial f}{\partial t} d^3\boldsymbol{v} + m \int v_a v_i \frac{\partial f}{\partial x_i} d^3\boldsymbol{v} + \int q v_a (E_i + \epsilon_{ijk} v_j B_k/c) \frac{\partial f}{\partial v_i} d^3\boldsymbol{v} \\
&= m \frac{\partial}{\partial t}(\int v_a f d^3\boldsymbol{v}) + m \frac{\partial}{\partial x_i}(\int v_a v_i f d^3\boldsymbol{v}) \\
&- qE_a(\int f d^3\boldsymbol{v}) - \frac{q}{m}\epsilon_{ajk}(\int v_j f d^3\boldsymbol{v}) B_k/c \\
&= \frac{\partial}{\partial t}(nmV_a) + \frac{\partial}{\partial x_i}(nmV_a V_i + p_{ai}) - q(E_a + \epsilon_{aij} V_i B_j/c) = 0
\end{aligned} \tag{8.10}$$

which is nothing but the motion (Euler) equation.

8.4 Dielectric tensor without external magnetic field

The equilibrium distribution function is an arbitrary function of the specific momentum $\boldsymbol{V} = \boldsymbol{p}/m$, $f_{eq} = n_0 f_0(\boldsymbol{V})$. Here we made the density explicit so that $\int f_0 d^3\boldsymbol{V} = 1$. In the equilibrium the charge density and the current density should vanish:

$$\sum n_0 q = 0 \tag{8.11}$$

$$\sum n_0 q \int \boldsymbol{v} f_0 d^3\boldsymbol{V} = 0 \tag{8.12}$$

The linearized equation for δf is

$$\frac{\partial}{\partial t}\delta f + \boldsymbol{v}\frac{\partial}{\partial \boldsymbol{r}}\delta f + \frac{q}{m}\left(\delta\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\delta\boldsymbol{B}\right)\frac{\partial}{\partial \boldsymbol{V}} f_0 = 0 \tag{8.13}$$

Fourier transforming and substituting $\delta\boldsymbol{B} = c\boldsymbol{k}\times\delta\boldsymbol{E}/\omega$, one gets

$$\delta f = -\frac{iq}{m}\left(\frac{\delta\boldsymbol{E}}{\omega} + \frac{\boldsymbol{k}(\boldsymbol{v}\delta\boldsymbol{E})}{\omega(\omega - \boldsymbol{k}\boldsymbol{v})}\right)\frac{\partial}{\partial \boldsymbol{V}} f_0 \tag{8.14}$$

$$\delta f = -\frac{iq}{m}\left(\frac{1}{\omega} + \frac{k_b v_a}{\omega(\omega - \boldsymbol{k}\boldsymbol{v})}\right)\left(\frac{\partial}{\partial u_a} f_0\right)\delta E_a \tag{8.15}$$

$$j_c = -\frac{iq^2n_0}{m}\int\left[\frac{v_c\delta_{ab}}{\omega} + \frac{k_b v_c v_a}{\omega(\omega - \boldsymbol{kv})}\right]\frac{\partial f_0}{\partial u_b}d^3\boldsymbol{V}\delta E_a \tag{8.16}$$

$$\sigma_{ca} = -\frac{iq^2n_0}{m}\int\left[\frac{v_c\delta_{ab}}{\omega} + \frac{k_b v_c v_a}{\omega(\omega - \boldsymbol{kv})}\right]\frac{\partial f_0}{\partial u_b}d^3\boldsymbol{V} \tag{8.17}$$

Let us choose the coordinates so that $\boldsymbol{k} = (0, 0, k)$, then

$$\sigma_{ca} = -\sum\frac{iq^2n_0}{m}\left[\int\frac{v_c}{\omega}\frac{\partial f_0}{\partial u_a}d^3\boldsymbol{V} + \int\frac{k_z v_c v_a}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V}\right] \tag{8.18}$$

$$\sigma_{xx} = -\sum\frac{iq^2n_0}{m}\left[\int\frac{v_x}{\omega}\frac{\partial f_0}{\partial u_x}d^3\boldsymbol{V} + \int\frac{k_z v_x^2}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V}\right] \tag{8.19}$$

$$\sigma_{yy} = -\sum\frac{iq^2n_0}{m}\left[\int\frac{v_y}{\omega}\frac{\partial f_0}{\partial u_y}d^3\boldsymbol{V} + \int\frac{k_z v_y^2}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V}\right] \tag{8.20}$$

$$\sigma_{xy} = -\sum\frac{iq^2n_0}{m}\left[\int\frac{v_x}{\omega}\frac{\partial f_0}{\partial u_y}d^3\boldsymbol{V} + \int\frac{k_z v_x v_y}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V}\right] \tag{8.21}$$

$$\sigma_{yx} = -\sum\frac{iq^2n_0}{m}\left[\int\frac{v_y}{\omega}\frac{\partial f_0}{\partial u_x}d^3\boldsymbol{V} + \int\frac{k_z v_x v_y}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V}\right] \tag{8.22}$$

$$\sigma_{xz} = -\sum\frac{iq^2n_0}{m}\left[\int\frac{v_x}{\omega}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V} + \int\frac{k_z v_x v_z}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V}\right] \tag{8.23}$$

$$\sigma_{zx} = -\sum\frac{iq^2n_0}{m}\left[\int\frac{v_z}{\omega}\frac{\partial f_0}{\partial u_x}d^3\boldsymbol{V} + \int\frac{k_z v_x v_z}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V}\right] \tag{8.24}$$

$$\sigma_{zz} = -\sum\frac{iq^2n_0}{m}\left[\int\frac{v_z}{\omega}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V} + \int\frac{k_z v_z^2}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V}\right] \tag{8.25}$$

Since $v_i = u_i/\gamma$, $\gamma^2 = 1 + u^2/c^2$, one has

$$\frac{\partial v_i}{\partial u_j} = \frac{\partial v_j}{\partial u_i} = \frac{\delta_{ij}}{\gamma} - \frac{u_i u_j}{c^2\gamma^3} \tag{8.26}$$

therefore $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, etc.

If the equilibrium distribution is even, $f_0(-\boldsymbol{V}) = f_0(\boldsymbol{V})$, then all integrals with odd powers of v_x and v_y vanish, and we have $\sigma_{xy} = \sigma_{yx} = 0$, $\sigma_{xz} = \sigma_{zx} = 0$, $\sigma_{yz} = \sigma_{zy} = 0$, and

$$\sigma_{xx} = \sum\frac{iq^2n_0}{m\omega}\int\frac{\partial v_x}{\partial u_x}f_0 d^3\boldsymbol{V} - \sum\frac{iq^2n_0}{m\omega}\int\frac{k_z v_x^2}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V} \tag{8.27}$$

$$\sigma_{yy} = \sum\frac{iq^2n_0}{m\omega}\int\frac{\partial v_y}{\partial u_y}f_0 d^3\boldsymbol{V} - \sum\frac{iq^2n_0}{m\omega}\int\frac{k_z v_y^2}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V} \tag{8.28}$$

$$\sigma_{zz} = -\sum\frac{iq^2n_0}{m\omega}\int\frac{v_z}{\omega - k_z v_z}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V} \tag{8.29}$$

For the dielectric tensor we now have

$$\epsilon_{aa} = 1 - \sum\frac{\omega_p^2}{\omega}\int\frac{1 - v_a^2/c^2}{\gamma}f_0 d^3\boldsymbol{V} + \sum\frac{\omega_p^2}{\omega}\int\frac{k_z v_a^2}{\omega(\omega - k_z v_z)}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V} \tag{8.30}$$

where $a = x, y$, and

$$\epsilon_{zz} = 1 + \sum\frac{\omega_p^2}{\omega}\int\frac{v_z}{\omega - k_z v_z}\frac{\partial f_0}{\partial u_z}d^3\boldsymbol{V} \tag{8.31}$$

It is easy to see that in a cold plasma, $f_0 = \delta(\boldsymbol{V})$, one arrives at $\epsilon_{ij} = (1 - \sum \omega_p^2/\omega^2)\delta_{ij}$.

8.5 Waves

In order to make this simple we study here only one-dimensional electrostatic wave without external magnetic field, that is, everything will depend only on z (and t), there will be only one, v_z component of the velocity, and only one, E_z component of the electric field. The corresponding Vlasov equation fill take the form

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} + \frac{q}{m} E_z \frac{\partial f}{\partial v_z} = 0. \tag{8.32}$$

Since we are going to study waves we have to start with the equilibrium, where $E_z = 0$, $(\partial f/\partial z) = 0$, and $(\partial f/\partial t) = 0$. Thus, the equilibrium distribution should depend only on v_z. We shall write $f_0(v_z) = nF_0(v_z)$, where n is the equilibrium density, and $\int F_0 dv_z = 1$. Perturbing, linearizing, and Fourier-transforming (8.32) one has

$$-i(\omega - kv_z)\tilde{f} = -\frac{qn}{m} E_z \frac{\partial F_0}{\partial v_z}. \tag{8.33}$$

Following general rules, we have to calculate current density, which in this case will be just

$$j_z = -\left(\sum_s \frac{i n_s q_s^2}{m_s} \int \frac{v_z}{\omega - kv_z} \frac{\partial F_{0s}}{\partial v_z} dv_z \right) E_z, \tag{8.34}$$

and, respectively,

$$\epsilon_{zz} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega} \int_{-\infty}^{\infty} \frac{v_z}{\omega - kv_z} \frac{\partial F_{0s}}{\partial v_z} dv_z. \tag{8.35}$$

Here $\omega_{ps}^2 = 4\pi n_s q_s^2/m_s$. The dispersion relation would read $\epsilon_{zz} = 0$.

8.6 Landau damping

The main task is to evaluate the integrals in (8.35). In order to do that we have to decide what to do with the singularity at $v_z = \omega/k$. The solution is the *adiabatic switch-on*, where a perturbation gradually exponentially increases from $t \to -\infty$, that is, $E_z \propto \exp(-i\omega t + \varepsilon t)$, with $\varepsilon \to +0$. Technically this means substitution $\omega \to \omega + i\varepsilon$, so that the singularity is removed from real v_z into the upper plane of the complex v_z. In other words, the integral over v_z in (8.35) is in fact a contour integral in the complex v_z plane, the contour running from $v_z = -\infty$ to $v_z = \infty$ *below* the singular point. The same can be symbolically written as follows:

$$\frac{1}{x + i\varepsilon} = \mathcal{P}\frac{1}{x} + i\pi\delta(x), \tag{8.36}$$

where $\mathcal{P}$ means the principal value integral

$$\mathcal{P}\int_a^b \frac{1}{x} F dx \equiv \lim_{\varepsilon\to 0}\left(\int_a^{-\varepsilon}\frac{1}{x}F dx + \int_\varepsilon^b \frac{1}{x} F dx\right),$$

and $\delta(x)$ is the usual delta-function. Summarizing all this, the dispersion relation for electrostatic waves can be written as follows

$$\begin{aligned}\epsilon_s = \frac{\omega_{ps}^2}{k^2}\Bigg[\mathcal{P}\int_{-\infty}^{\infty}\frac{1}{v_z - \omega/k}\frac{\partial F_{0s}}{\partial v_z} dv_z \\ + i\pi \frac{\partial F_0}{\partial v_z}|_{v_z=\omega/k}\Bigg] = 0,\end{aligned} \tag{8.37}$$

where $\epsilon_{zz} = 1 + \sum_s \epsilon_s$.

The singularity $v_z = \omega/k$ is called *Cerenkov resonance.* Since now the dispersion relation is complex its solution ω should be also complex. Let $\omega \to \omega + i\Gamma$, where we retain notation ω for the real part (frequency). Then $E \propto \exp(-i\omega t + \Gamma t)$. If $\Gamma < 0$ the wave amplitude decreases with time, that is, the wave is damped. We shall see immediately that this is the situation in the plasma in a thermodynamic equilibrium, where the distribution functions have the form (Maxwellian)

$$F_0 = \frac{1}{\sqrt{2\pi} v_T}\exp(-v_z^2/2v_T^2). \tag{8.38}$$

Here v_T is the thermal velocity related to the species temperature as follows: $v_T^2 = T/m$.

The integral in (8.37) cannot be calculated analytically in the whole range of ω/k. Maximum of the integrand $\frac{\partial F}{\partial v_z}$ is in the range $v_z \sim v_T$, so that it is reasonable to expect that far from this region some approximations would be useful.

High phase-velocity. In this case $\omega/k \gg v_T$ and we expand

$$\frac{1}{\omega/k - v_z} = \frac{k}{\omega}\left[1 + \frac{k v_z}{\omega} + \frac{k^2 v_z^2}{\omega}^2 + \dots\right].$$

Substituting and integrating we find

$$\operatorname{Re}\epsilon_s = -\frac{\omega_{ps}^2}{\omega^2}\left(1 + \frac{3k^2 v_T^2}{\omega^2}\right). \tag{8.39}$$

Low phase-velocity. In this case $\omega/k \ll v_T$. We introduce a shift $v_z = u + \omega/k$, then

$$\begin{aligned}F_0 &= \frac{1}{\sqrt{2\pi} v_T}\exp\left(-\frac{\omega^2}{2k^2 v_T^2} - \frac{\omega u}{2k v_T^2} - \frac{u^2}{2v_T^2}\right) \\ &= \frac{1}{\sqrt{2\pi} v_T}\exp\left(-\frac{\omega^2}{2k^2 v_T^2}\right)\left[1 - \frac{\omega u}{2k v_T^2} - -\frac{u^2}{2v_T^2} + \dots\right]\end{aligned} \tag{8.40}$$

Since $(\partial F_0/\partial v_z) dv_z = (\partial F_0/\partial u) du$, one finally finds

$$\operatorname{Re}\epsilon_s = \frac{\omega_{ps}^2}{k^2 v_T^2}. \tag{8.41}$$

Imaginary part. The imaginary part in both cases is

$$\operatorname{Im}\epsilon_s = \frac{\sqrt{\pi}\omega_{ps}^2\omega}{\sqrt{2}k^3 v_{Ts}^3}\exp(-\omega^2/2k^2v_{Ts}^2). \tag{8.42}$$

Now we are ready to investigate electrostatic waves in an electron-ion plasma, where $m_i/m_e \geq 2000 \gg 1$. We also assume that $T_i \leq T_e$, so that $v_{Te}/v_{Ti} = (T_e/T_i)^{1/2}(m_i/m_e)^{1/2} \gg 1$. We shall analyze separately three regions: (a) $\omega/k \gg v_{Te}$, (b) $v_{Te} \gg \omega/k \gg v_{Ti}$, and (c) $v_{Ti} \gg \omega/k$. We are looking for waves with $|\Gamma| \ll \omega$ (otherwise we cannot speak about a wave). The dispersion relation is developed as follows:

$$\epsilon(\omega + i\Gamma) = 0, \tag{8.43}$$

$$\operatorname{Re}\epsilon(\omega + i\Gamma) + i\operatorname{Im}\epsilon(\omega + i\Gamma) = 0, \tag{8.44}$$

$$\operatorname{Re}\epsilon(\omega) + i\Gamma\frac{\partial}{\partial\omega}\operatorname{Re}\epsilon(\omega) + i\operatorname{Im}\epsilon(\omega) = 0, \tag{8.45}$$

$$\operatorname{Re}\epsilon(\omega) = 0, \tag{8.46}$$

$$\Gamma = -\operatorname{Im}\epsilon(\omega)\left[\frac{\partial}{\partial\omega}\operatorname{Re}\epsilon(\omega)\right]^{-1}. \tag{8.47}$$

In other words, ω is found from (8.46), without taking into account the imaginary part of the dielectric tensor. The growth (damping) rate Γ is then found from (8.47) where we should substitute ω which was found earlier.

High phase velocity range: Langmuir waves. In the range $\omega/k \gg v_{Te} \gg v_{Ti}$ we have (8.39) for electrons and ions as well, so that

$$\operatorname{Re}\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2}(1 + \frac{3k^2v_{Te}^2}{\omega^2}) = 0, \tag{8.48}$$

where we neglected the ion contribution. The dispersion relation now reads

$$\omega^2 = \omega_{pe}^2 + 3k^2v_{Te}^2, \tag{8.49}$$

and describes Langmuir waves with thermal effects taken into account. For the imaginary part we have

$$\frac{\partial}{\partial\omega}\operatorname{Re}\epsilon(\omega) \approx \frac{2}{\omega_{pe}}$$

and

$$\begin{aligned}\frac{\Gamma}{\omega_{pe}} &= -\frac{\pi^{1/2}\omega_{pe}^3}{2^{3/2}k^3v_{Te}^3}\exp(-\omega_{pe}^2/2k^2v_{Te}^2)\\ &= -\frac{\pi^{1/2}}{2^{3/2}k^3r_{De}^3}\exp(-1/2k^2r_{De}^2)\end{aligned} \tag{8.50}$$

where $r_{De} = v_{Te}/\omega_{pe} = \sqrt{T_e/4\pi ne^2}$ is the electron Debye radius. According to our condition $kr_{De} \ll 1$, and $\omega \approx \omega_{pe}$, so that $|\Gamma|/\omega \ll 1$, that is, Langmuir waves are weakly damped.

Intermediate phase velocity range. In the range $v_{Te} \gg \omega/k \gg v_{Ti}$ we have

$$\mathrm{Re}\,\epsilon_e = \frac{1}{k^2 r_{De}^2}, \qquad \mathrm{Re}\,\epsilon_i = -\frac{\omega_{pi}^2}{\omega^2},$$

so that we have

$$\mathrm{Re}\,\epsilon = 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{1}{k^2 r_{De}^2} = 0, \tag{8.51}$$

from which we get the dispersion relation for *ion-sound* waves:

$$\omega^2 = \frac{\omega_{pi}^2 k^2 r_{De}^2}{1 + k^2 r_{De}^2}. \tag{8.52}$$

In the limit $kr_{De} \ll 1$ we have

$$\omega = kv_{Te}\omega_{pi}/\omega_{pe} = k\sqrt{T_e/m_i}. \tag{8.53}$$

In the limit $kr_{De} \gg 1$ we get $\omega \to \omega_{pi}$. Since we required $v_{Te} \gg \omega/k \gg v_{Ti}$ we have the condition $T_e \gg T_i$.

Low phase velocities. In the range $\omega/k \ll v_{Te}, v_{Ti}$ one has

$$\mathrm{Re}\,\epsilon = 1 + \frac{1}{k^2 r_{De}^2} + \frac{1}{k^2 r_{Di}^2} = 0, \tag{8.54}$$

and we rediscover Debye screening $k = \pm i1/r_D$, where $r_D^{-2} = r_{De}^{-2} + r_{Di}^{-2}$.

8.7 Problems

Problem 8.1. Calculate damping rate for ion-sound waves.

Problem 8.2. Derive dispersion relations for cold plasmas using $F_0 = \delta(v)$.

Problem 8.3. Find the dispersion relation for waves in electron-positron plasma with the "waterbag" distribution: $F_0 = \theta(v_0^2 - v^2)/2v_0$, where $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ if $x < 0$.

Problem 8.4. Derive the dispersion relation for electromagnetic waves ($\boldsymbol{E} \perp \boldsymbol{k}$, $\boldsymbol{B} \neq 0$).

Problem 8.5. Derive the dispersion relation for electrostatic waves in a plasma consisting of cold electrons and ions, moving with the relative velocity V_0.

Chapter 9

Micro-instabilities

In this chapter we learn about instabilities (spontaneous growth of perturbations) caused by highly nonequilibrium distributions of plasma.

9.1 Beam (two-stream) instability

Let us consider a plasma consisting of two electron populations: body $f_p = n_p\delta(v)$, and beam $f_b = n_b\delta(v - V_0)$. We shall assume that the beam density is much smaller than the plasma body density, $n_b/n_p \ll 1$. It is easy to find

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - kV_0)^2}. \tag{9.1}$$

where $\omega_p^2 = 4\pi n_p e^2/m$ and $\omega_b^2 = 4\pi n_b e^2/m$. It is worth noting that $\omega - kV_0$ is just the frequency, Doppler shifted into the beam rest frame.

The dispersion relation reads

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - kV_0)^2} = 0. \tag{9.2}$$

In the high frequency limit, $\omega \gg kV_0$, one immediately has $\omega = \omega_p$ (Langmuir wave). There is no low frequency limit $\omega \ll kV_0$. If $|\omega - kV_0| \ll kV_0$ one has

$$\omega = kV_0 \pm \omega_b \left[1 - \frac{\omega_p^2}{k^2V_0^2}\right]^{-1/2} \tag{9.3}$$

When $kV_0 < \omega_p$ the square root becomes imaginary. For long wave lengths $kV_0 \ll \omega_p$ one gets the *growth rate*

$$\Gamma = \operatorname{Im}\omega = \pm ikV_0(n_b/n_p)^{1/2}$$

The positive solution gives the instability: the wave amplitude grows exponentially. The real part of the dispersion relation $\omega = kV_0$ shows that the growing oscillations have zero frequency it the beam frame.

When $kV_0 \approx \omega_p$ the growth rate goes to infinity and more accurate consideration is necessary. In this region the beam waves $\omega = kV_0$ can be in the resonance with the plasma body (Langmuir) waves $\omega = \omega_p$. Putting $kV_0 = \omega_p$, $\omega = \omega_p + \delta$, one gets

$$\frac{2\delta}{\omega_p} = \frac{\omega_b^2}{\delta^2}$$

and the unstable solution reads

$$\delta = 2^{1/3}\omega_p(n_b/n_p)^{1/3}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \tag{9.4}$$

9.2 More on the beam instability

Let us consider again the dispersion relation for the beam instability,

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - kV_0)^2}.$$

If there were only the plasma body, $\omega_b = 0$, we would have regular plasma waves $\omega = \pm\omega_p$, where $\pm$ stands to show that the are waves propagating in both directions. The energy of this wave is, according to (6.36),

$$U_p = \frac{2\omega_p^2}{\omega^2}\frac{|E_p|^2}{8\pi},$$

and is always positive.

If there were only beam particles, $\omega_p = 0$, then we would have two beam waves with the dispersion relation

$$\omega = kV_0 \pm \omega_b,$$

with the wave energy

$$U_b = \frac{2\omega_p^2\omega}{(\omega - kV_0)^3}\frac{|E_b|^2}{8\pi}.$$

The slower wave $\omega = kV_0 - \omega_b$ has *negative* energy. It should be understood that this energy is negative in the plasma frame. In the beam frame the wave energy remains positive.

Let us now come back to the plasma-beam system. If two waves with the opposite sign of energy resonate (couple) so that they have close k and ω, electric field in both can grow while maintaining energy conservation. Thus, we can expect that instability occurs where $kV_0 - \omega_b \approx \omega_p$. The condition corresponds to the above resonant hydrodynamic beam instability.

9.3 Bump-on-tail instability

Let us now consider the electrostatic instability of a thermal plasma, that is, we assume that there is a region in the velocity space where $dF_0/dv > 0$. Following the prescriptions outlined in the chapter on kinetic description, we find the growth rate in the form

$$\gamma = -\frac{\operatorname{Im}\epsilon}{\partial \operatorname{Re}\epsilon/\partial\omega},$$

where

$$\operatorname{Im}\epsilon = -\pi\frac{\omega_p^2}{k|k|}\frac{dF_0}{dv}|_{v=\omega/k}$$

and

$$\frac{\partial \operatorname{Re}\epsilon}{\partial \omega} = -\frac{\omega_p^2}{k^3}\mathcal{P}\int_{-\infty}^{\infty}\frac{1}{(v-\omega/k)^2}\frac{dF_0}{dv}dv.$$

It can be shown that the above expression may be written as

$$\frac{\partial \operatorname{Re}\epsilon}{\partial \omega} = \frac{2}{k(v_{ph}-v_g)},$$

where $v_{ph} = \omega/k$ and $v_g = d\omega/dk$. Eventually,

$$\gamma = \pi\frac{\omega_p^2(v_{ph}-v_g)}{|k|}\frac{dF_0}{dv}|_{v=\omega/k}. \tag{9.5}$$

Let us assume that $\frac{dF_0}{dv} > 0$ for $v_1 < v < v_2$. The meaning of this *kinetic* instability is that a number of modes with $v_1 < \omega/k < v_2$ are excited. Lets have a closer look at the unstable mode with the wave-vector k. The electric field in this mode has the time dependence of the form

$$E_k(t) \propto \exp(-i\omega_k t + \gamma_k t)$$

Chapter 10

Nonlinear phenomena

Appendix A

Plasma parameters

Table A.1: Parameters of various plasmas

Source	Density, cm^{-3}	Temperature, K	Composition	Magnetic field, T
Solar wind near Earth	1-10	10^5	p,e	10 nT
Fusion reactor	10^{15}	10^8		
Ionosphere	10^5	500	7	
Glow discharge	10^9	10^4		
Flame	10^8	10^3		
Interplanetary plasma	1	100		