The Bertrand theorem revisited

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The Bertrand theorem, which states that the only power-law central potentials for which the bounded trajectories are closed are $1/r^2$ and r^2, is analyzed using the Poincaré–Lindstedt perturbation method. This perturbation method does not generate secular terms and correctly incorporates the effect of nonlinearities on the nature of periodic solutions. The requirement that the orbits be closed implies that the theorem holds in each order of the expansion.
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I. INTRODUCTION

The Bertrand theorem states that the only power-law central potentials for which the bounded trajectories are closed are the $1/r$ (gravitational and Coulomb) and r^2 (isotropic three-dimensional harmonic oscillator).[1] Proofs of this theorem can be found in many textbooks on classical mechanics (see, for example, Refs. 2–5).

Planar motion is described by two degrees of freedom: r, the distance of the orbiting particle from the origin, and φ, the angle between the position radius-vector and some initial direction. Due to the conservation of angular momentum, the two equations of motion can be reduced to a single one:

$$\frac{d^2u}{d\varphi^2}+u=-\frac{\mu}{L^2}\frac{dV}{du}\equiv J(u)\quad (u=1/r), \tag{1}$$

where V is the central potential. The term u on the left-hand side of Eq. (1) arises from the centrifugal potential. The constants on the right-hand side are μ, the reduced mass, and L, the (conserved) magnitude of the angular momentum.

Under appropriate conditions, Eq. (1) describes oscillations of the radius, r, between two extreme values. The period of the oscillations is the angular sector, $\Delta\varphi$, covered by the orbiting point as the radius completes one full cycle between its smallest and largest values. For a bounded orbit to be closed, $\Delta\varphi$ must be a rational fraction of 2π. With m and n integers, we must have

$$\Delta\varphi=\frac{n}{m}2\pi. \tag{2}$$

This condition ensures that the oscillatory motion of the radius completes m full cycles when the angle φ completes n revolutions in the plane.

Circular motion corresponds to a constant radius, r_0, the value of which is given by

$$u_0=J(u_0),\quad (u_0=1/r_0). \tag{3}$$

If the orbit is not circular, then $u\neq u_0$. The analysis is then carried out in terms of x, the deviation of u from u_0:

$$x=u-u_0,\quad \varepsilon\equiv\max(|x|)/u_0\ll 1. \tag{4}$$

In nonlinear oscillatory systems, the period varies with the amplitude of the oscillations. Because Eq. (1) is a nonlinear equation for $u(\varphi)$, one expects $\Delta\varphi$ to vary with the amplitude of x. Hence, Eq. (2) is bound to be violated, unless the effect of the nonlinearity on $\Delta\varphi$ is eliminated. This idea is the fundamental ingredient in all proofs of the theorem.

In one typical proof, $\Delta\varphi$ is computed in the extreme limits of small and large values of x, which correspond to small and large eccentricity, respectively. The requirement that $\Delta\varphi$ has the same value in both limits and satisfies Eq. (2) yields the result of the theorem (see, for example, Refs. 4 and 5).

Another proof employs a perturbation expansion in powers of x for small x (see, for example, Ref. 3)

$$J(u)=J(u_0)+\sum_{n\geq 1}J_n x^n,\quad J_n=\left.\frac{d^nJ(u)}{du^n}\right|_{u=u_0}. \tag{5}$$

Equation (1) leads to

$$\frac{d^2x}{d\varphi^2}+\omega_0^2\,x=\sum_{n\geq 2}J_n\,x^n,$$
$$\omega_0^2=1-J_1=1-\left.\frac{dJ(u)}{du}\right|_{u=u_0}=1+\frac{\mu}{L^2}\left.\frac{d^2V}{du^2}\right|_{u=u_0}. \tag{6}$$

If we omit the nonlinear perturbation on the right-hand side of Eq. (6), the unperturbed motion is oscillatory, provided that

$$\omega_0^2>0. \tag{7}$$

To ensure that the orbit of the actual planar motion is closed, Eq. (2) must be obeyed, leading to

$$\omega_0^2=(m/n)^2. \tag{8}$$

If we introduce the nonlinear perturbation, the angular velocity, ω, deviates from its unperturbed value, ω_0, and varies continuously as the magnitude of x is increased. Hence, as the amplitude is varied, ω goes through infinitely many irrational values. Therefore, for most values of the amplitude, *the orbit will not be closed*. To ensure that the orbit is closed, one must require that ω remains equal to ω_0 *despite* the effect of the nonlinearity.

We now expand x in a perturbative series

$$x(\varphi)=\sum_{k\geq 1}a_k\cos(k\omega_0\varphi). \tag{9}$$

The magnitude of a_k is expected to decrease as k increases, because they are generated in progressively higher orders of the perturbation expansion. The requirement that ω_0 is a rational number yields the result of the theorem.

However, the expansion of Eq. (9), if employed in a consistent manner in a perturbation analysis, yields unbounded (secular) terms, which are also aperiodic. These terms make the perturbation expansion useless beyond short times of $O(1)$. Moreover, they generate an aperiodic approximation to a physical system, whose motion is periodic. This well-known problem and its, equally well known, resolution, will be discussed in Sec. II.

II. POINCARÉ–LINDSTEDT EXPANSION

The emergence of secular terms is avoided if instead of a "naïve" expansion, one uses the method of Poincaré[6] and Lindstedt.[7] (In Sec. 28 of Ref. 8, the method is used in the analysis of anharmonic oscillations.) The method is described in detail in Refs. 9 and 10. Identical results are obtained in the methods of multiple-time-scales,[11–13] normal forms,[14–17] and averaging.[18] The essence of the approach is the fact that the nonlinear perturbation on the right-hand side of Eq. (6) changes the angular velocity, ω_0, into an *updated* angular velocity, ω.

The natural variable for analyzing Eq. (6) is $\tau = \omega\varphi$, rather than $\omega_0\varphi$. In terms of τ, Eq. (6) becomes

$$\omega^2 \frac{d^2x}{d\tau^2} + \omega_0^2 x = \sum_{n\geqslant 2} J_n x^n. \tag{10}$$

Because Eq. (10) describes the motion of an energy conserving system, the solution is expected to be periodic in τ.

We now expand *both* the solution, x, and the angular velocity, ω, in a power series in the small parameter ε [see Eq. (4)]:

$$x = \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots), \tag{11}$$

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3 + \cdots. \tag{12}$$

In the naïve method, all the ω_n, $n \geqslant 1$ are set to zero. Hence, the angular velocity is *not updated*. With this choice, the solution is expanded in terms of functions whose period coincides with the unperturbed one. The undesired consequences of this choice are discussed below.

If we substitute Eqs. (11) and (12) in Eq. (10) and require that the resulting equation holds in every order separately, we obtain through third order in ε,

$$\frac{d^2x_0}{d\tau^2} + x_0 = 0, \tag{13}$$

$$\frac{d^2x_1}{d\tau^2} + x_1 = -2\frac{\omega_1}{\omega_0}\frac{d^2x_0}{d\tau^2} + \frac{1}{2\omega_0^2} J_2 x_0^2, \tag{14}$$

$$\frac{d^2x_2}{d\tau^2} + x_2 = -2\frac{\omega_2}{\omega_0}\frac{d^2x_0}{d\tau^2} - \frac{\omega_1^2}{\omega_0^2}\frac{d^2x_0}{d\tau^2} - 2\frac{\omega_1}{\omega_0}\frac{d^2x_1}{d\tau^2} + \frac{1}{\omega_0^2} J_2 x_0 x_1 + \frac{1}{6\omega_0^2} J_3 x_0^3, \tag{15}$$

$$\frac{d^2x_3}{d\tau^2} + x_3 = -2\frac{\omega_3}{\omega_0}\frac{d^2x_0}{d\tau^2} - 2\frac{\omega_1\omega_2}{\omega_0^2}\frac{d^2x_0}{d\tau^2} - \frac{\omega_1^2}{\omega_0^2}\frac{d^2x_1}{d\tau^2} - 2\frac{\omega_2}{\omega_0}\frac{d^2x_1}{d\tau^2} - 2\frac{\omega_1}{\omega_0}\frac{d^2x_2}{d\tau^2} + \frac{1}{2\omega_0^2} J_2 x_1^2 + \frac{1}{\omega_0^2} J_2 x_0 x_2 + \frac{1}{2\omega_0^2} J_3 x_0^2 x_1 + \frac{1}{24\omega_0^2} J_4 x_0^4. \tag{16}$$

The coefficients ω_n will be chosen so as to guarantee that the solution is bounded and periodic in τ. Equation (13) is solved by

$$x_0 = a_0 \cos(\tau + \tau_0). \tag{17}$$

As the starting point does not affect the analysis, $\tau_0 = 0$ is used in the following discussion. We substitute Eq. (17) in Eq. (14) and obtain

$$\frac{d^2x_1}{d\tau^2} + x_1 = 2\frac{\omega_1}{\omega_0} a_0 \cos\tau + \frac{J_2 a_0^2}{4\omega_0^2}(1 + \cos 2\tau). \tag{18}$$

The homogeneous part of Eq. (18) describes the motion of a harmonic oscillator with angular velocity equal to unity. The $\cos\tau$ term on the right-hand side has the same angular velocity. Hence, it is a *resonant* term. If retained, it would contribute to the solution of Eq. (18) an aperiodic term of the form

$$\frac{\omega_1}{\omega_0} a_0 \tau \sin\tau.$$

This term is unbounded as a function of τ, and is a *secular* term. As the solution is expected to be periodic in τ, such a term would provide a poor approximation for the solution. Hence, one must have

$$\omega_1 = 0. \tag{19}$$

Thus, in this order of the expansion, the naïve choice is the correct one.

Now that the resonant term has been removed, Eq. (18) can be solved, yielding a periodic solution

$$x_1 = a_{11}\cos\tau + a_{12}\sin\tau + \frac{a_0^2 J_2}{4\omega_0^2} - \frac{a_0^2 J_2}{12\omega_0^2}\cos 2\tau. \tag{20}$$

If we use Eqs. (17) and (20) in Eq. (15), the latter becomes

$$\frac{d^2x_2}{d\tau^2} + x_2 = \left(2\frac{\omega_2}{\omega_0} a_0 + \frac{5J_2^2 a_0^3}{24\omega_0^4} + \frac{J_3 a_0^3}{8\omega_0^2}\right)\cos\tau + \frac{J_2 a_0 a_{11}}{2\omega_0^2} + \frac{J_2 a_0 a_{11}}{2\omega_0^2}\cos 2\tau + \frac{J_2 a_0 a_{12}}{2\omega_0^2}\sin 2\tau + \left(\frac{J_3 a_0^3}{24\omega_0^2} - \frac{J_2^2 a_0^3}{24\omega_0^4}\right)\cos 3\tau. \tag{21}$$

The "naïve" choice of $\omega_2 = 0$ leaves the resonant term (proportional to $\cos\tau$) intact, the result being that the solution of Eq. (21) for x_2 contains a term that is proportional to τ. Such an unbounded secular term spoils the periodic nature of the approximate solution, and renders the perturbation expansion useless beyond short times of $O(1)$. Thus, the $\cos\tau$ term must be eliminated, leading to

$$\omega_2=-(5J_2^2+3J_3\omega_0^2)\,\frac{a_0^2}{48\omega_0^3}. \tag{22}$$

Equation (21) may now be solved for x_2, yielding a solution that is periodic in τ,

$$x_2=a_{21}\cos\tau+a_{22}\sin\tau+\frac{a_0a_{11}J_2}{2\omega_0^2}-\frac{a_0a_{11}J_2}{6\omega_0^2}\cos 2\tau -\frac{a_0a_{12}J_2}{6\omega_0^2}\sin 2\tau+\frac{(J_2^2-J_3\omega_0^2)}{192\omega_0^4}\cos 3\tau. \tag{23}$$

The procedure can be repeated in higher orders. Elimination of resonant terms in each order determines the coefficients ω_n in the expansion of ω [Eq. (12)], thereby ensuring that the approximate solution is bounded and periodic in τ. The results for the updating of the angular velocity are given below through $O(\varepsilon^4)$,

$$\omega_3=-(5J_2^2+3J_3\omega_0^2)\frac{a_0a_{11}}{24\omega_0^3}, \tag{24}$$

$$\omega_4=-\left(\frac{2a_0a_{21}+a_{11}^2+a_{12}^2}{48\omega_0^3}+\frac{19a_0^4J_3}{5760\omega_0^5}\right)(5J_2^2+3J_3\omega_0^2) -\frac{97a_0^4}{13\,824\omega_0^7}(5J_2^2+3J_3\omega_0^2)^2-\frac{7a_0^4J_3^2}{480\omega_0^3}-\frac{7a_0^4J_2J_4}{384\omega_0^3} -\frac{a_0^4J_5}{384\omega_0}. \tag{25}$$

III. BERTRAND THEOREM

The nonlinear term in Eq. (10) generates corrections to the frequency. These corrections depend on the amplitude in a continuous manner. Consequently, even if ω_0 is rational, the updated angular velocity, ω, will be irrational in general. For ω to assume rational values for all amplitudes, it must remain equal to ω_0. Hence, all corrections that make ω different from ω_0 must vanish,

$$\omega_2=\omega_3=\omega_4=\cdots=0. \tag{26}$$

In other words, one is seeking central potentials for which, despite the nonlinearities that they generate, no resonant terms appear in any order of the perturbative expansion. For such potentials, the naïve expansion does hold.

If we apply Eq. (26) to Eq. (22), we obtain

$$(5J_2^2+3J_3\omega_0^2)=0. \tag{27}$$

Equation (27) ensures that ω_3 also vanishes, and simplifies the requirement that ω_4 vanishes,

$$28J_3^2+35J_2J_4+5J_5\omega_0^2=0. \tag{28}$$

To see what power-law central potentials are allowed by the requirements of Eqs. (8), (27), and (28), we write $V(r)$ as

$$V(r)=\alpha r^s=\alpha u^{-s}. \tag{29}$$

If we use the definition of $J(u)$ [see Eq. (1)], this form leads to

$$J(u)=\frac{\alpha s\mu}{L^2}u^{-s-1}. \tag{30}$$

Equation (4) then yields

$$u_0=\left(\frac{\alpha s\mu}{L^2}\right)^{1/(s+2)}. \tag{31}$$

The derivatives J_n are readily computed and given here through $n=5$,

$$J_1=-(s+1), \tag{32}$$

$$J_2=\frac{(s+1)(s+2)}{u_0}, \tag{33}$$

$$J_3=-\frac{(s+1)(s+2)(s+3)}{u_0^2}, \tag{34}$$

$$J_4=\frac{(s+1)(s+2)(s+3)(s+4)}{u_0^3}, \tag{35}$$

$$J_5=-\frac{(s+1)(s+2)(s+3)(s+4)(s+5)}{u_0^4}. \tag{36}$$

If we substitute Eqs. (31) and (32) in Eq. (8), we find

$$\omega_0^2=s+2=\left(\frac{m}{n}\right)^2. \tag{36}$$

We require that ω_2 and ω_4 vanish and find that the only solutions common to both are

$$s=-2,-1,+2. \tag{37}$$

(The $\omega_4=0$ equation has additional solutions. However, they are not solutions of the requirement that $\omega_2=0$.) The same conclusion is reached in higher orders. The case $s=-2$ corresponds to $\omega_0=0$, namely, no oscillatory behavior, and hence is excluded. As a result, the only power-law central potentials that admit closed orbits are the $1/r$ and the r^2 potentials.

From the viewpoint of perturbation theory, the Bertrand theorem has the following significance. In general, the nonlinearities generated by a central potential in Eq. (10) lead to the appearance of resonant terms in the dynamical equations in all, or most, orders of the expansion. To avoid the emergence of unbounded and aperiodic *secular* terms in the perturbative expansion, one must account for the *update* in the angular velocity, ω. The only power-law central potentials for which no resonant terms emerge in the perturbative expansion are the two unique potentials singled out by the theorem. Because these potentials do not generate resonant terms, there is also no updating of the angular velocity, and the naïve expansion yields the correct results.

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Celestial Globe. This Celestial Globe at Oberlin College is by Gilman Joslin (1804-ca.1886). Joslin worked in many fields in addition to making globes: he was one of the first Americans to make a daguerreotype and was engaged in shipbuilding. The celestial globe showed the stars forming the constellations and also the mythical figures associated with them. (D. J. Warner, ''Geography of Heaven and Earth—III,'' Rittenhouse, 2, (1988), pp. 88–89) (Photograph and notes by Thomas B. Greenslade, Jr., Kenyon College)